

THE K-THEORY OF LEFT POINTED DERIVATORS

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ABSTRACT. We build on work of Muro-Raptis in [MR17] and Cisinski-Neeman in [CN08] to prove that the additivity of derivator K-theory holds for a large class of derivators that we call *left pointed derivators*, which includes all triangulated derivators. The proof methodology is an adaptation of the combinatorial methods of Grayson in [Gra11]. As a corollary, we prove that derivator K-theory is an infinite loop space. Finally, we speculate on the role of derivator K-theory as a trace from the algebraic K-theory of a stable ∞ -category à la [BGT13].

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1. INTRODUCTION

Algebraic K-theory is a general tool for understanding complicated mathematical objects arising in homotopy theory, algebraic geometry, differential topology, representation theory, and other fields. Since Quillen's seminal work [Qui73], the field of algebraic K-theory has enjoyed incredible popularity and expansion beyond abelian or exact categories. Waldhausen in [Wal85] set the tone for how K-theory would be constructed for more and more general objects. A common philosophy is that, if we expand the class of objects on which K-theory is defined, we should make sure that our new definitions agree with the old. Waldhausen made sure this was the case when he defined his K-theory of categories with cofibrations and weak equivalences.

A stumbling block, however, was including triangulated categories into Waldhausen's framework. Defined first by Verdier in his doctoral thesis [Ver96] in 1963,

triangulated categories are invaluable in the study of homological algebra and homotopy theory. The bounded derived category associated to an exact or abelian category has a natural triangulated structure. The homotopy categories of both ordinary and G -equivariant spectra are similarly naturally triangulated. In algebraic geometry, the theory of motives is studied using triangulated categories, as the ‘abelian category of mixed motives’ remains a conjecture.

Neeman in the 1990’s published a series of papers on the K-theory of triangulated categories starting with [Nee97a] and [Nee97b]. There are a number of interesting properties of his construction, but we do not mention them here due to a fundamental defect in triangulated category theory. In the years following Neeman’s publications, various authors proved that a satisfactory functorial construction on triangulated categories would never be possible, in the following sense. Starting from an exact category, we can take its K-theory via Quillen’s or Waldhausen’s definition, or pass to its bounded derived category and take its triangulated K-theory. If triangulated K-theory were to extend Quillen’s K-theory, these two constructions would give the same K-groups. In other words, if K_Q is Quillen’s K-theory and \mathcal{S} is the category of spaces, can we find a functor K_Δ making the diagram below commute?

$$\begin{array}{ccc}
 \mathbf{ExCat} & \xrightarrow{K_Q} & \mathcal{S} \\
 & \searrow^{D^b(-)} & \nearrow^{K_\Delta} \\
 & \mathbf{TriCat} &
 \end{array}$$

Schlichting in [Sch02] gives a general argument showing that we should not expect the triangulated category to retain all K-theoretic information. Specifically, he constructs two Waldhausen categories \mathcal{W}_1 and \mathcal{W}_2 , arising from abelian categories of modules over a commutative ring, such that their (triangulated) homotopy categories are equivalent but their Waldhausen K_4 differs. However, on the triangulated side, each homotopy category $\mathrm{Ho}(\mathcal{W}_i)$ appears as a Verdier localization of the same category with equivalent localising subcategories. This leads to a contradiction between two desirable properties for K-theory: agreement and localization. The above diagram can only commute if K_Δ does not satisfy localization.

Schlichting’s result points to the need for a richer structure for homotopy theory than triangulated categories alone. This is not a new idea; from our first homological algebra class, we learn that the cone construction in a triangulated category is non-functorial. The slogan ‘unique up to unique isomorphism’, central to how we approach category theory, abandons us. There are a few different ways to give ourselves more data to work with.

In some senses, the best replacement for triangulated categories, especially through the lens of algebraic K-theory, are *stable ∞ -categories*, i.e. the triangulated analogue of higher categories. Recent work of Blumberg-Gepner-Tabuada in [BGT13] proves

that algebraic K-theory is the universal additive invariant of a stable (small) ∞ -category. This extends the origins of K-theory precisely; the Grothendieck group of a commutative monoid is the universal abelian group such that any additive invariant on the monoid must factor through it. However, there are reasons to mistrust ∞ -categories: the literature is daunting and there are competing (though equivalent) models. Though the theory of ∞ -categories is ideal for universal constructions, it is often difficult to know concretely what has been constructed. In the example of algebraic K-theory, there is an ∞ -category of ‘non-commutative motives’ in which algebraic K-theory is found – but the rest of the category is quite mysterious (for now – there is much active work on this topic).

There are lower-categorical tools that work well and do not have these drawbacks. An early tool in studying triangulated categories, developed by Quillen in [Qui67] before his work on algebraic K-theory, is that of *model categories*. A model category is the data of a category we wish to treat homotopically and extra information allowing us to pass from the rigid structure to the homotopy category. A solution to the non-functoriality of the cone can be solved in such a framework. For nice enough model categories \mathcal{M} , the category of arrows $\text{Ar } \mathcal{M}$ inherits a compatible model category structure. We can define the cone of a morphism before passing to the homotopy categories, i.e. $\text{Ho}(\text{Ar } \mathcal{M}) \rightarrow \text{Ho}(\mathcal{M})$ rather than $\text{Ar } \text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{M})$. If we knew only the category $\text{Ho}(\mathcal{M})$, this first approach will not be possible, so in this sense we have given ourselves more to work with.

Let us interpret this in the triangulated setting. Let \mathcal{A} be an abelian category. Then the arrow category $\text{Ar } \mathcal{A}$ is still abelian, so we can take its bounded derived category $D^b(\text{Ar } \mathcal{A})$, which we can think of as homotopy classes of maps of chain complexes. Then we can define the cone construction as an exact functor of abelian categories $C^b(\text{Ar } \mathcal{A}) \rightarrow C^b(\mathcal{A})$ *before* we invert quasi-isomorphisms. We still have a functor upon passing to the derived category, and so have a functorial cone construction with a new domain. However, there is a forgetful functor $D^b(\text{Ar } \mathcal{A}) \rightarrow \text{Ar } D^b(\mathcal{A})$ which takes a homotopy class of a map to a map of homotopy classes. While $D^b(\text{Ar } \mathcal{A})$ is triangulated, $\text{Ar } D^b(\mathcal{A})$ is not, but this functor can be shown to be full and essentially surjective (and almost never faithful). We have constructed a cone *functor* because we had access to \mathcal{A} itself and not just the triangulated category $D^b(\mathcal{A})$ and thus were able to build an auxiliary diagram category $\text{Ar } \mathcal{A}$ to fill in the gaps in information.

This is our slogan: we would like to study not only a triangulated category, but a whole system of triangulated diagram categories. An equivalence of homotopy categories as in [Sch02] does not necessarily give rise to an equivalence of systems, and thus we are able to better distinguish distinct homotopy theories. Grothendieck in [Gro90] coined the term *derivator* for a system of derived categories, and this is

the framework in which we will work to address questions about the K-theory of triangulated categories.

The theory of derivators was developed initially (under different names) by Heller in [Hel88], Grothendieck in [Gro90], and (in the triangulated setting) Franke in [Fra96]. In brief, a derivator represents an abstract bicomplete homotopy theory; we attach the adjective *triangulated* to a derivator when it represents a stable (bicomplete) homotopy theory. The fundamental proof techniques used in the theory of derivators and the diagrammatic flavor which is unique to this field have been well-articulated by Moritz Rahn (né Groth) in [Gro13].

The K-theory of triangulated derivators was defined by Maltiniotis in [Mal07] and Garkusha in [Gar06] and [Gar05], and revisited by Muro and Raptis in [MR17]. Muro-Raptis proved that the definition of K-theory still makes sense for derivators which are not triangulated, and form a class which we call *left pointed derivators*. We develop in [Col20] a more robust theory of such *half derivators*, i.e. ones representing homotopy theories that may not be bicomplete, but still admit many limits or colimits, in order to answer questions about K-theory in the broadest generality. This is one advantage over the approach of [BGT13]: we are not restricted to stable phenomena.

Cisinski and Neeman proved that the K-theory of triangulated derivators satisfies a form of additivity in [CN08], but their proof involves Neeman's theory of regions and does not admit an obvious analogy in the non-triangulated situation. We prove the following broader theorem.

Main Result. (Theorems 5.5 and 5.20)

Let \mathbb{D} and \mathbb{E} be left pointed derivators. Then the following are equivalent and true:

- (1) The map

$$\mathbb{D}_{\text{cof}} \xrightarrow{(0,0)^* \times (1,1)^*} \mathbb{D} \times \mathbb{D}$$

induces a homotopy equivalence on derivator K-theory, where \mathbb{D}_{cof} is the left pointed derivator of cofiber sequences in \mathbb{D} .

- (2) If $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$ is a cofibration morphism of derivators, then there exists a homotopy between the target of the cofibration morphism and the source plus quotient. Specifically,

$$K(T) \simeq K(S) \sqcup K(Q) (\cong K(S \sqcup Q))$$

where \sqcup denotes the coproduct.

The first statement is the form of additivity proven by Cisinski-Neeman and conjectured by Maltiniotis. The second statement is the one proven in Theorem 5.20 and uses techniques of Grayson in [Gra11] that have a more diagrammatic flavor appropriate for general derivator theory. We obtain as a corollary the delooping of

the K-theory space $K(\mathbb{D})$, and so conclude that $K(\mathbb{D})$ is an infinite loop space for a general left pointed derivator \mathbb{D} , which was not known in any cases before. This answers two questions of Muro-Raptis posed in [MR17].

We first recall the necessary results from [Col20] that establish the domain of derivator K-theory. We then give the construction of derivator K-theory and present previously-known results. We conclude with the main new additivity theorem of derivator K-theory and the important consequences thereof.

2. PRELIMINARIES

Recall that a *prederivator* is just a strict 2-functor $\mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$, where the domain is the 2-category of small categories and the codomain the ‘2-category’ of not-necessarily-small categories. For a morphism $u: J \rightarrow K$ in \mathbf{Cat} we denote by u^* the functor $\mathbb{D}(u): \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ in \mathbf{CAT} , and for $\alpha: u \Rightarrow v$ in \mathbf{Cat} we denote by α^* the natural transformation $\mathbb{D}(\alpha): u^* \Rightarrow v^*$. Composition is respected strictly, so that $(vu)^* = u^*v^*$ and $(\alpha \odot \beta)^* = \alpha^* \odot \beta^*$ (here \odot is the pasting of natural transformations). Identities are also preserved, so that $(\text{id}_J)^* = \text{id}_{\mathbb{D}(J)}$ and $(\text{id}_u)^* = \text{id}_{u^*}$.

A *derivator* is a prederivator that models a system of diagram categories which is homotopically bicomplete. We give the definition in two parts.

Let \mathbb{D} be a prederivator, K a small category, and $k \in K$ be any object. Recall that we have a functor that classifies the object k which we denote $k: e \rightarrow K$, where e is the category with one object and one (identity) morphism. Then for any $X \in \mathbb{D}(K)$, we have an object $k^*X \in \mathbb{D}(e)$. Suppose that $f: k_1 \rightarrow k_2$ is a map in K . Then we have a corresponding natural transformation $f^*: k_1^* \Rightarrow k_2^*$ and thus a map $f^*X: k_1^*X \rightarrow k_2^*X$ in $\mathbb{D}(e)$. Repeating this process for all objects and maps in K , we obtain a functor

$$\text{dia}_K: \mathbb{D}(K) \rightarrow \text{Fun}(K, \mathbb{D}(e))$$

which sends $X \in \mathbb{D}(K)$ to the functor which assembles all the above data. We call this an *underlying diagram functor*, and its existence implies that the prederivator \mathbb{D} should be modelling K -shaped diagrams in $\mathbb{D}(e)$, which we call the *underlying category* or the *base* of the prederivator. We will refer to the categories $\mathbb{D}(K)$ as *coherent* diagrams, as opposed to the *incoherent* diagrams $\text{Fun}(K, \mathbb{D}(e))$.

Definition 2.1. A *semiderivator* is a prederivator \mathbb{D} satisfying the following two axioms:

(Der1) Coproducts are sent to products. Explicitly, consider any set $\{K_a\}_{a \in A}$ of small categories, and let $i_b: K_b \rightarrow \coprod_{a \in A} K_a$ be the inclusion for any $b \in A$.

Pulling back along this inclusion gives a functor

$$i_b^*: \mathbb{D} \left(\coprod_{a \in A} K_a \right) \rightarrow \mathbb{D}(K_b)$$

which induces a map to the product

$$\prod_{b \in A} i_b^*: \mathbb{D} \left(\coprod_{a \in A} K_a \right) \rightarrow \prod_{b \in A} \mathbb{D}(K_b)$$

We require this map to be an equivalence of categories for any collection $\{K_a\}_{a \in A}$.

(Der2) Isomorphisms are detected pointwise. That is, for any $K \in \mathbf{Cat}$, the underlying diagram functor dia_K is conservative. More specifically, a map $f: X \rightarrow Y$ is an isomorphism in $\mathbb{D}(K)$ if and only if the map $k^*f: k^*X \rightarrow k^*Y$ is an isomorphism for all $k \in K$.

These two axioms comprise the ‘system of diagram categories’ part of the definition. For the next two axioms, we need the following notation.

Definition 2.2. Let $u: J \rightarrow K$ be any functor, and let $k \in K$ be any object. We define the *comma category* (u/k) as follows: its objects are pairs $j \in J$ with a map $f: u(j) \rightarrow k$, and a map $(j, f) \rightarrow (j', f')$ in the comma category is a map $g: j \rightarrow j'$ in J making the obvious diagram commute:

$$(2.3) \quad \begin{array}{ccc} u(j) & \xrightarrow{u(g)} & u(j') \\ & \searrow f & \swarrow f' \\ & & k \end{array}$$

For any category $K \in \mathbf{Cat}$, we write π_K for the unique functor $K \rightarrow e$.

Definition 2.4. A semiderivator \mathbb{D} is a *left derivator* if it satisfies the following two axioms:

(Der3L) The base of the semiderivator $\mathbb{D}(e)$ is (homotopically) cocomplete. Specifically, for every functor $u: J \rightarrow K$, the pullback u^* admits a left adjoint, which we denote $u_!: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ and call the *(homotopy) left Kan extension along u* . As a special case, this includes $\pi_K: K \rightarrow e$ and thus $\mathbb{D}(e)$ admits all (coherent) colimits.

(Der4L) Left Kan extensions can be computed pointwise. Let $u: J \rightarrow K$ and $k \in K$. Then we have the following lax pullback square in **Cat**:

$$\begin{array}{ccc} (u/k) & \xrightarrow{\text{pr}} & J \\ \pi \downarrow & \alpha \swarrow & \downarrow u \\ e & \xrightarrow{k} & K \end{array}$$

where we let $\pi = \pi_{(u/k)}$ for brevity. Applying the semiderivator \mathbb{D} to this square, we obtain the following square in **CAT**, remembering that functors are reversed and natural transformations are not:

$$\begin{array}{ccc} \mathbb{D}((u/k)) & \xleftarrow{\text{pr}^*} & \mathbb{D}(J) \\ \pi^* \uparrow & \alpha^* \swarrow & \uparrow u^* \\ \mathbb{D}(e) & \xleftarrow{k^*} & \mathbb{D}(K) \end{array}$$

By Der3L, both vertical functors admit left adjoints, so we may construct the left mate of α^* as the pasting of the below diagram, which we denote by $\alpha_!$ (rather than the ‘official’ notation $(\alpha^*)_!$):

$$\begin{array}{ccccccc} \mathbb{D}(e) & \xleftarrow{\pi_!} & \mathbb{D}((u/k)) & \xleftarrow{\text{pr}^*} & \mathbb{D}(J) & \xleftarrow{=} & \\ & \swarrow & \uparrow \pi^* & \swarrow \alpha^* & \uparrow u^* & \swarrow & \\ & & \mathbb{D}(e) & \xleftarrow{k^*} & \mathbb{D}(K) & \xleftarrow{u_!} & \mathbb{D}(J) \end{array}$$

In total we have the natural transformation $\alpha_!: \pi_! \text{pr}^* \Rightarrow k^* u_!$. We require this map to be a natural isomorphism.

A semiderivator \mathbb{D} is a *right derivator* if it satisfies the analogous axioms Der3R and Der4R, which together say that every functor u^* admits a right adjoint u_* satisfying a pointwise computation formula. A *derivator* is just a left and right derivator.

There is a relative construction that we need to introduce at this point. Suppose that \mathbb{D} is a prederivator and $I \in \mathbf{Cat}$ is a category. Then we can define another prederivator \mathbb{D}^I by $\mathbb{D}^I(K) = \mathbb{D}(I \times K)$; for $u: J \rightarrow K$, $\mathbb{D}^I(u) = \mathbb{D}(\text{id}_I \times u)$; and similar for natural transformations. If \mathbb{D} is a (left/right/full) derivator, so is \mathbb{D}^I . This is often called a *shifted derivator*.

There is a ‘fifth axiom’ for derivators that is not needed in all contexts, but is needed for ours.

Definition 2.5. A prederivator is *strong* if for any finite free category I and for any category $K \in \mathbf{Cat}$, the partial underlying diagram functor

$$\mathrm{dia}_{K,I}: \mathbb{D}(K \times I) \rightarrow \mathrm{Fun}(K, \mathbb{D}(I))$$

is full and essentially surjective. In some derivator literature this axiom is called Der5.

The functor $\mathrm{dia}_{K,I}$ is related to the underlying diagram functor defined above, except in this case we leave the I -dimension of all coherent diagrams intact. The content of this axiom is that any incoherent diagram of a simple shape is liftable to a coherent one; further, any map of these incoherent diagrams lifts to a map between the coherent ones. This axiom is asking for the same sort of thing as lifting a map between objects in some homotopy category to a map of bifibrant replacements.

Remark 2.6. The strongness axiom usually only asks for the case $I = [1]$ (and sometimes all finite ordinals $[n]$, see Notation 3.1), but all known examples either satisfy this ‘strong strongness’ version of the axiom or fail for $I = [1]$. Derivators failing the case $I = [1]$ are constructed in [LN17], so any version of strongness is a non-extraneous axiom, but all derivators arising from some sort of model satisfy Der5 as above (see Lemma 3.8 below).

We have one more adjective to attach to our derivators.

Definition 2.7. A derivator \mathbb{D} is *pointed* if its underlying category $\mathbb{D}(e)$ is pointed, i.e. the unique morphism from the initial to the final object is an isomorphism. We will write $0 \in \mathbb{D}(e)$ for its zero object.

This definition is easy to check but does not tell the whole story. One immediate consequence is that each category $\mathbb{D}(J)$ has a zero object, given by $0_J := \pi_J^*(0)$ where $\pi_J: J \rightarrow e$ is the projection. The more interesting corollary requires some definitions first:

Definition 2.8. Let $u: J \rightarrow K$ be a fully faithful functor that is injective on objects.

- (1) The functor u is a *sieve* if for any morphism $k \rightarrow u(j)$ in K , k lies in the image of u .
- (2) The functor u is a *cosieve* if for any morphism $u(j) \rightarrow k$ in K , k lies in the image of u .

Proposition 2.9 (Proposition 1.23, [Gro13]). Let \mathbb{D} be a pointed derivator, and let $u: J \rightarrow K$ be a sieve (resp. cosieve). Then $u_*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ (resp. $u_!$) is fully faithful, with essential image $X \in \mathbb{D}(K)$ such that $k^*X \cong 0$ for all $k \in K \setminus u(J)$.

These adjoints are called *extension by zero morphisms* and are essential in the proofs of derivator K-theory below.

Proposition 2.10 (Corollaries 3.5 and 3.8, [Gro13]). A derivator is pointed if and only if extension by zero morphisms admit exceptional adjoints. Specifically, every right extension by zero u_* along a sieve admits a right adjoint $u^!$ and every left extension by zero $u_!$ along a cosieve admits a left adjoint $u^?$.

The existence of exceptional adjoints is crucial for functoriality properties on derivator K-theory, which we now address.

Definition 2.11. Let $\mathbb{D}, \mathbb{E}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ be prederivators. A *morphism of prederivators* $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ is a pseudonatural transformation of the associated 2-functors. This consists of the following data: for each $K \in \mathbf{Cat}$ we have a functor $\Phi_K: \mathbb{D}(K) \rightarrow \mathbb{E}(K)$ and for every $u: J \rightarrow K$ we have a natural isomorphism $\gamma_u^\Phi: u^* \Phi_K \Rightarrow \Phi_J u^*$

$$\begin{array}{ccc} \mathbb{D}(K) & \xrightarrow{\Phi_K} & \mathbb{E}(K) \\ u^* \downarrow & \gamma_u^\Phi \swarrow & \downarrow u^* \\ \mathbb{D}(J) & \xrightarrow{\Phi_J} & \mathbb{E}(J) \end{array}$$

where we have slightly abused notation by writing u^* for both $\mathbb{D}(u)$ and $\mathbb{E}(u)$. These are subject to certain coherence conditions which we leave to [Col20], [Gro13], or [Bor94, §7].

When defining morphisms of derivators, instead of writing $\Phi_K(X)$ for $X \in \mathbb{D}(K)$ for all $K \in \mathbf{Cat}$, we will usually write ΦX for $X \in \mathbb{D}$. Our constructions will be not heavily dependent on specific K .

A morphism of derivators is just a morphism of the underlying prederivators. We say that a morphism of (pre)derivators is an *equivalence* if each functor Φ_K is an equivalence of categories. There is a subclass of morphisms that need singling out.

Definition 2.12. A morphism of (pre)derivators $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ is called *strict* if for every $u: J \rightarrow K$, the structure isomorphism $\gamma_u^\Phi: u^* \Phi_K \Rightarrow \Phi_J u^*$ is the identity.

In 2-categorical language, a strict morphism Φ is a *strict* natural transformation of 2-functors, not just *pseudonatural*. A morphism being strict seems fairly unlikely, as it implies a great deal of rigidity in what is a fairly flexible homotopical context. Nonetheless, the model of derivator K-theory we use in this paper will require strict morphisms, and we will be able to obtain strict morphisms (up to equivalence) whenever we need.

The main class of morphisms of derivators that we study involve shifted derivators. Suppose that $u: J \rightarrow K$ is a functor and \mathbb{D} is a prederivator. Then we obtain a morphism of prederivators $u^*: \mathbb{D}^K \rightarrow \mathbb{D}^J$ which is actually strict, as the coherence data γ^{u^*} arise from the strict 2-functoriality of $\mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$. Moreover, if \mathbb{D}

is a left or right derivator, we obtain morphisms $u_!, u_*: \mathbb{D}^J \rightarrow \mathbb{D}^K$, but these are *not* strict. This is related to the fact that (co)limits are essentially unique, which allows for the construction of the structure isomorphisms, but not actually unique. These morphisms enjoy other properties which we will describe now.

Definition 2.13. Let \mathbb{D}, \mathbb{E} be left derivators and $u: J \rightarrow K$ in \mathbf{Cat} . We say that a morphism $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ *preserves left Kan extensions along u* if the left mate of $(\gamma_u^\Phi)^{-1}$ is a natural isomorphism. Specifically, we have the pasting

$$\begin{array}{ccccc}
 \mathbb{D}(J) & \xrightarrow{u_!} & \mathbb{D}(K) & \xrightarrow{\Phi_K} & \mathbb{E}(K) \\
 \searrow & \nearrow & \downarrow u^* & \nearrow (\gamma_u^\Phi)^{-1} & \downarrow u^* \\
 & & \mathbb{D}(J) & \xrightarrow{\Phi_J} & \mathbb{E}(J) \\
 & & \searrow & \nearrow & \downarrow u_! \\
 & & & & \mathbb{E}(K)
 \end{array}
 \quad \begin{array}{l}
 \text{= (top)} \\
 \text{= (bottom)} \\
 \text{= (left)} \\
 \text{= (right)} \\
 \text{= (outer)}
 \end{array}$$

giving us a natural transformation $(\gamma_u^\Phi)_!^{-1}: u_! \Phi_J \Rightarrow \Phi_K u_!$ which we demand is an isomorphism, where again we slightly abuse notation by writing $u_!$ for the left adjoint to both $\mathbb{D}(u)$ and $\mathbb{E}(u)$. If the morphism Φ preserves left Kan extensions along all $u: J \rightarrow K$ in \mathbf{Cat} , we say that Φ is *cocontinuous*.

There is an analogous notion of *continuous morphism* that we will not spell out (as we will not need it below).

Cocontinuous morphisms of derivators can appear in the same way as colimit-preserving functors in category theory: via adjunctions.

Definition 2.14. Given two morphisms of (pre)derivators $\Phi, \Psi: \mathbb{D} \rightarrow \mathbb{E}$, a natural transformation $\rho: \Phi \Rightarrow \Psi$ is given by a *modification* of pseudonatural transformations. This is the data of a natural transformation $\rho_K: \Phi_K \Rightarrow \Psi_K$ for every $K \in \mathbf{Cat}$ satisfying coherence conditions that we do not record here.

Definition 2.15. Let $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ and $\Psi: \mathbb{E} \rightarrow \mathbb{D}$ be two morphisms of (pre)derivators. We say that Φ is *left adjoint to Ψ* (equivalently, Ψ is *right adjoint to Φ*) if there exist two modifications $\eta: \text{id}_{\mathbb{D}} \Rightarrow \Psi\Phi$ and $\varepsilon: \Phi\Psi \Rightarrow \text{id}_{\mathbb{E}}$ satisfying the usual triangle identities.

In particular, an adjunction of morphisms of derivators (Φ, Ψ) gives rise to an adjunction of functors (Φ_K, Ψ_K) for each $K \in \mathbf{Cat}$. However, this condition is not sufficient. A morphism of derivators $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ may admit a right adjoint to $\Phi_K: \mathbb{D}(K) \rightarrow \mathbb{E}(K)$ for all $K \in \mathbf{Cat}$, but part of the data of a right adjoint morphism of derivators is the structure isomorphisms, which we have no way of recovering in this general situation.

Lemma 2.16 (Proposition 2.9, [Gro13]). Let $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ be a morphism of left derivators such that each Φ_K admits a right adjoint Ψ_K . Then the collection of

functors $\{\Psi_K\}$ assembles to a morphism of derivators $\Psi: \mathbb{E} \rightarrow \mathbb{D}$ which is right adjoint to Φ if and only if Φ is cocontinuous.

The morphism Φ being cocontinuous allows us to construct the coherence isomorphisms γ_u^Ψ and ‘glue together’ the various Ψ_K . We are not claiming anything like an adjoint functor theorem for general derivators, so this lemma does not admit a converse.

There are two classes of examples that give us everything we need for this paper. Let $u: J \rightarrow K$ be a functor in **Cat**. If u admits a categorical right adjoint $v: K \rightarrow J$, then $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ is right adjoint to $v^*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ because (strict) 2-functors send adjunctions to adjunctions, though in our case which is left and which is right swaps. We can upgrade this to, for any prederivator \mathbb{D} , a (cocontinuous) left adjoint morphism $v^*: \mathbb{D}^K \rightarrow \mathbb{D}^J$ which preserves any left Kan extensions that \mathbb{D}^K happens to have.

If \mathbb{D} is a left derivator, then the left adjoint functor $u_!: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ lifts to a left adjoint morphism of derivators $u_!: \mathbb{D}^J \rightarrow \mathbb{D}^K$ with right adjoint u^* . Similarly, if \mathbb{D} is a right derivator, $u_*: \mathbb{D}^J \rightarrow \mathbb{D}^K$ is a right adjoint morphism of derivators with left adjoint u^* . In fact, for any prederivator \mathbb{D} , the morphism $u^*: \mathbb{D}^K \rightarrow \mathbb{D}^J$ preserves all left and right Kan extensions that \mathbb{D} happens to have by [Gro13, Proposition 2.5] for categorical reasons.

Finally, suppose \mathbb{D} is a pointed derivator and $u: J \rightarrow K$ is a sieve. Then the right extension by zero $u_*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ admits an exceptional right adjoint by Proposition 2.10, so u_* is a left adjoint and hence cocontinuous.

3. LEFT POINTED DERIVATORS

Having set up the basic vocabulary of the theory of derivators, we can begin to examine what we actually need for K-theory.

To motivate the following definition, we recall the definition of K_0 of an abelian category \mathcal{A} . It is constructed as the free abelian group on (isomorphism classes of) objects $A \in \mathcal{A}$, written $[A] \in K_0(\mathcal{A})$, under the relation that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence, we have $[B] = [A] + [C]$. A short exact sequence is equivalently a cocartesian square

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ 0 & \rightarrow & C \end{array}$$

under the assumption that $A \rightarrow B$ is a monomorphism. Thus if we are to construct even K_0 for a derivator, it needs to admit a notion of (coherent) cocartesian squares and a zero object.

Notation 3.1. For $n \in \mathbb{N}$, let $[n]$ denote the totally ordered set with $n + 1$ elements:

$$0 \rightarrow 1 \rightarrow \cdots \rightarrow n - 1 \rightarrow n$$

Each of these are finite free categories.

Notation 3.2. Let \square be the category $[1] \times [1]$, with labelling

$$\begin{array}{ccc} (0, 0) & \rightarrow & (1, 0) \\ \downarrow & & \downarrow \\ (0, 1) & \rightarrow & (1, 1) \end{array}$$

Let $i_\Gamma : \Gamma \rightarrow \square$ be the full subcategory of \square lacking the element $(1, 1)$.

Definition 3.3. Let \mathbb{D} be a left derivator and $X \in \mathbb{D}(\square)$. We say that X is *co-cartesian* (i.e. a pushout square) if X is in the essential image of $i_{\Gamma,!} : \mathbb{D}(\Gamma) \rightarrow \mathbb{D}(\square)$. Otherwise put, X is *cocartesian* if the counit $i_{\Gamma,!} i_\Gamma^* X \rightarrow X$ of the $(i_{\Gamma,!}, i_\Gamma^*)$ adjunction is an isomorphism.

This is where Muro and Raptis obtained their domain for derivator K-theory: they considered left derivators which admit a zero object. However, would like to be able to *construct* pushouts appropriate for computing K_0 as above. This means coherently making cocartesian squares starting from an element in $\mathbb{D}(\Gamma)$ of the form

$$(3.4) \quad \begin{array}{ccc} & a \rightarrow b & \\ & \downarrow & \\ & 0 & \end{array}$$

In order to construct this Γ -shaped diagram starting from a coherent arrow $(a \rightarrow b) \in \mathbb{D}([1])$, we need more than the structure of a left derivator.

Definition 3.5. A prederivator $\mathbb{D} : \mathbf{Dia}^{\text{op}} \rightarrow \mathbf{CAT}$ is a *left pointed derivator* if it is a strong left derivator, $\mathbb{D}(e)$ is pointed, and for every sieve $u : J \rightarrow K$, u^* admits a right adjoint u_* satisfying Der4R.

Indeed, the inclusion $i_{[1]} : [1] \rightarrow \Gamma$ is a sieve, so by Proposition 2.9 we can compute that

$$(3.6) \quad (a \xrightarrow{f} b) \xrightarrow{i_{[1],*}} \begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \\ 0 & & \end{array} \xrightarrow{i_{\Gamma,!}} \begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ 0 & \rightarrow & C(f) \end{array}$$

where we have named the object at the $(1, 1)$ position the *cone* of the (coherent) morphism $f \in \mathbb{D}([1])$. This means that there is a morphism of derivators $\mathbb{D}^{[1]} \rightarrow \mathbb{D}^\square$ realising the above diagram.

Remark 3.7. Given that we can lift incoherent diagrams in the shape of finite free categories, it should be pointed out that Γ is such a shape. Therefore since we can build diagrams of shape Diagram 3.4 incoherently, we can lift them to coherent objects of $\mathbb{D}(\Gamma)$. From that point we can take the coherent pushout via $i_{\Gamma,!}$. We cannot lift ‘incoherent pushout squares’ because \square is not finite free.

However, this process spoils any hope of functoriality in the construction of the coherent pushout of a morphism starting from $\mathbb{D}([1])$, and this functoriality is essential. The requirement that our left pointed derivators be strong is used only to check the the computation at Diagram 5.17; it requires using ‘incoherent reasoning’ that must be lifted up to the derivator and does not interfere with any functoriality.

It may be possible that the computation can be made without strongness, but this author does not have a proof. It may also be that this computation *requires* strongness, and a proof is also lacking for this possibility. This small point does not take away from the main result of the paper, so we leave it for future consideration.

The key example of a left pointed derivator, and indeed the motivation of the abstract definition, is the following, drawn from Corollary 2.24, Proposition 3.4, and Lemma 4.3 in [Cis10].

Lemma 3.8. Let \mathcal{W} be a saturated Waldhausen category satisfying the cylinder axiom. Then the associated prederivator $\mathbb{D}_{\mathcal{W}}: K \mapsto \text{Ho}(\text{Fun}(K, \mathcal{W}))$ defined on $\mathbf{Dir}_{\mathbf{f}}$ is a (strong) left pointed derivator. Moreover, an exact functor of Waldhausen categories induces a cocontinuous morphism of the corresponding derivators. In particular, these morphisms preserve cocartesian squares and the zero object.

Recall that derivators need not be defined on all of \mathbf{Cat} , but on sub-2-categories $\mathbf{Dia} \subset \mathbf{Cat}$ satisfying some closure properties. One key example is $\mathbf{Dir}_{\mathbf{f}}$, which consists of all *finite direct categories*, i.e. categories whose nerve has only finitely many nondegenerate simplices. These are also called *homotopy finite categories* by [Arl20] and [GPS14]. General Waldhausen categories will not admit arbitrary colimits and will admit no (non-empty) limits whatsoever.

For the purposes of K-theory, $\mathbf{Dir}_{\mathbf{f}}$ is an ideal domain for our left pointed derivators. Homotopical cocompleteness for all of \mathbf{Cat} means the existence of infinite coproducts. This allows for a derivator version of the usual Eilenberg swindle on K-theory, e.g. [Wei13, V.1.9.1]. Since this trick requires additivity, we will prove it below as a corollary of the main theorem at Proposition 6.1.

Hereafter we let \mathbf{Der}_K be the 1-category with objects strong left pointed derivators on $\mathbf{Dir}_{\mathbf{f}}$ and morphisms cocontinuous morphisms of derivators up to invertible modification. That is, we consider $\Phi, \Psi: \mathbb{D} \rightarrow \mathbb{E}$ to be the same if there exists a zig-zag of invertible modifications from Φ to Ψ . We do this because such morphisms will induce homotopic maps on K-theory, as we will show in Corollary 4.2 shortly. We will leave the adjective ‘strong’ implicit throughout.

4. DERIVATOR K-THEORY

It is helpful at this point to recall Waldhausen's K-theory for a category with cofibrations and weak equivalences from [Wal85]. To such a category \mathcal{W} we assign a simplicial object in Waldhausen categories $\mathbf{S}_\bullet \mathcal{W}$, where $\mathbf{S}_n \mathcal{W}$ is the category of exact functors from the arrow category $\text{Ar}[n]$ to \mathcal{W} . Taking the wide subcategory with only maps the weak equivalences $w\mathbf{S}_\bullet \mathcal{W}$, we obtain again a simplicial Waldhausen category. Then we define K-theory as follows:

$$K(\mathcal{W}) := \Omega|N_\bullet w\mathbf{S}_\bullet \mathcal{W}|,$$

the loop space of the (diagonal) geometric realisation of the bisimplicial set given by the nerve.

In [MR17], Muro and Raptis improved upon a construction of Garkusha in [Gar05] which generalises Waldhausen's \mathbf{S}_\bullet construction. First, we can restate the \mathbf{S}_\bullet construction in the language of derivators. To help with notation, for a category $[n] \in \Delta$, let the elements of its arrow category $\text{Ar}[n]$ be written (i, j) for $i \rightarrow j$.

Let \mathbb{D} be a left pointed derivator. We let $\mathbf{S}_n \mathbb{D}$ be the full subcategory of $\mathbb{D}(\text{Ar}[n])$ of objects X such that:

- (1) For every $0 \leq i \leq n$, $(i, i)^* X \in \mathbb{D}(e)$ is a zero object.
- (2) For every fully faithful inclusion $\iota: \square \rightarrow \text{Ar}[n]$, the object $\iota^* X \in \mathbb{D}(\square)$ is cocartesian.

In (2), it suffices to check only the inclusions such that $\iota(0, 1) = (i, i)$ by [Gro13, Proposition 3.13]. We then define *derivator K-theory* by

$$K(\mathbb{D}) = \Omega|N_\bullet i\mathbf{S}_\bullet \mathbb{D}|$$

where $i\mathbf{S}_n \mathbb{D} \subset \mathbf{S}_n \mathbb{D}$ is the wide subcategory consisting only of isomorphisms, in analogy with $w\mathbf{S}_n \mathcal{C}$, and the geometric realization is taking diagonally.

To give a few examples, first we have that $\mathbf{S}_0 \mathbb{D} \subset \mathbb{D}(\text{Ar}[0]) = \mathbb{D}(e)$ is trivial: it has only one object and property (1) above requires it to be a zero object of $\mathbb{D}(e)$. The category $\mathbf{S}_1 \mathbb{D} \subset \mathbb{D}(\text{Ar}[1])$ is slightly more interesting: it is a staircase with one nontrivial object:

$$\begin{array}{ccc} 0 & \rightarrow & a \\ & & \downarrow \\ & & 0 \end{array}$$

There are no fully faithful inclusions $\square \rightarrow \text{Ar}[1]$, so there is nothing else to require. We can see that $\mathbf{S}_1 \mathbb{D} = \mathbb{D}(e)$ as a category, an observation we will use later. The

first interesting category is $\mathbf{S}_2\mathbb{D} \subset \mathbb{D}(\text{Ar}[2])$, whose objects have the form

$$\begin{array}{ccc} 0 & \rightarrow & a \xrightarrow{f} b \\ & & \downarrow & \downarrow g \\ & & 0 & \rightarrow & c \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

with the requirement that the square is cocartesian. This is where we see $K_0(\mathbb{D})$ being encoded: the zero simplices of $K(\mathbb{D}) = \Omega|N_\bullet i\mathbf{S}_\bullet\mathbb{D}|$ come from $\mathbf{S}_1\mathbb{D}$ (because Ω gives a dimension shift) and these zero simplices are identified in $\pi_0 K(\mathbb{D})$ due to the existence of a path, i.e. an element of $\mathbf{S}_2\mathbb{D}$, relating them. Thus three objects appearing a cocartesian square in $\mathbb{D}(\square)$ leads to a relation on the homotopy classes of zero simplices in $\pi_0 K(\mathbb{D})$. The same thing happens at each $\pi_n K(\mathbb{D})$, but the relationship is more difficult to describe for large values of n .

We said above that in order for a morphism $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ to induce a map on K-theory, it needs to preserve cocartesian squares and the zero object, which we could then conclude was equivalent to asking for Φ to be cocontinuous. However, there is another problem. If we have a cocontinuous morphism $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ that is not *strict*, then Φ may not induce a map of simplicial sets $\mathbf{S}_\bullet\mathbb{D} \rightarrow \mathbf{S}_\bullet\mathbb{E}$. If we take (for example) the face map $d_i: [n] \rightarrow [n+1]$, then we obtain a diagram

$$\begin{array}{ccc} \mathbf{S}_{n+1}\mathbb{D} & \xrightarrow{\Phi_{\text{Ar}[n+1]}} & \mathbf{S}_{n+1}\mathbb{E} \\ d_i^* \downarrow & \gamma_{d_i}^\Phi \swarrow & \downarrow d_i^* \\ \mathbf{S}_n\mathbb{D} & \xrightarrow{\Phi_{\text{Ar}[n]}} & \mathbf{S}_n\mathbb{E} \end{array}$$

But this diagram commutes only up to natural isomorphism, so Φ does not give us an honest natural transformation of the bisimplicial sets $N_\bullet i\mathbf{S}_\bullet\mathbb{D} \rightarrow N_\bullet i\mathbf{S}_\bullet\mathbb{E}$. However, we have the following proposition to aid us.

Proposition 4.1 (Proposition 10.14, [CN08]). Let $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ be a morphism of prederivators. Then there exists a prederivator $\widetilde{\mathbb{D}}$, a strict equivalence of derivators $\Pi_\Phi: \widetilde{\mathbb{D}} \rightarrow \mathbb{D}$, and a strict morphism $\widetilde{\Phi}: \widetilde{\mathbb{D}} \rightarrow \mathbb{E}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \widetilde{\mathbb{D}} & \\ \Pi_\Phi \swarrow & & \searrow \widetilde{\Phi} \\ \mathbb{D} & \xrightarrow{\Phi} & \mathbb{E} \end{array}$$

~

We name the equivalence Π_Φ because it is some sort of projection, though we will not need the precise formula. If Φ is cocontinuous, $\widetilde{\Phi}$ is also cocontinuous because

it is the composition of cocontinuous morphisms. The following corollary is found at [CN08, Corollary 10.19] or [MR17, Remark 5.1.4].

Corollary 4.2. Any cocontinuous morphism of left pointed derivators $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ gives rise to a map on derivator K-theory $K(\Phi): K(\mathbb{D}) \rightarrow K(\mathbb{E})$ in \mathcal{S} , the homotopy category of spaces. Moreover, this association is functorial, in the sense that we have a 1-functor $\mathbf{Der}_K \rightarrow \mathcal{S}$ after inverting isomodifications in \mathbf{Der}_K to obtain a 1-category.

An immediate consequence is that equivalent left pointed derivators have equivalent K-theories.

There are some first results that are worth collecting. Maltiniotis in [Mal07] proved that, if \mathbb{D} is a triangulated derivator, then $K_0(\mathbb{D}(e))$ of the underlying triangulated category is equivalent to $K_0(\mathbb{D})$. He also established a comparison map from Quillen's K-theory of an exact category to derivator K-theory, which was subsequently extended to Waldhausen categories by Garkusha in [Gar06].

Specifically, let \mathcal{W} be a saturated Waldhausen category satisfying the cylinder axiom as in Lemma 3.8 so that it gives rise to a left pointed derivator $\mathbb{D}_{\mathcal{W}}$ (hereafter we will leave these assumptions on \mathcal{W} implicit). Then we have an obvious map

$$\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{W}) \rightarrow \mathbb{D}_{\mathcal{W}}(\mathrm{Ar}[n]) = \mathrm{Ho}(\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{W}))$$

sending a diagram to its homotopy class. This map restricts to $\mathbf{S}_n \mathcal{W} \rightarrow \mathbf{S}_n \mathbb{D}_{\mathcal{W}}$, sends weak equivalences to isomorphisms, and behaves well with respect to the simplicial structures, so we obtain a map of spaces

$$\mu: K(\mathcal{W}) \rightarrow K(\mathbb{D}_{\mathcal{W}})$$

Maltiniotis' proof is easily rewritten to imply that $\mu_0 := \pi_0 \mu$ is an isomorphism, and Muro in [Mur08] proved that μ_1 is an isomorphism as well. Muro's techniques are a bit ad hoc, but recent work of Raptis [Rap19, Theorem 5.5] proves that the comparison map μ is 2-connected, recovering Muro's result on μ_1 and proving that μ_2 is surjective. Maltiniotis conjectured that μ should be a weak homotopy equivalence in the case that $\mathbb{D}_{\mathcal{W}}$ is triangulated, and the same question can be asked in general.

Unfortunately, the conjecture fails almost totally. Muro and Raptis together in [MR11] show that μ will generally not be an equivalence for triangulated derivators arising from stable module categories. In that same work, the authors use the example of Schlichting in [Sch02] to prove that any K-theory of derivators invariant under equivalences of derivators cannot satisfy both agreement and send Verdier localizations of triangulated derivators to homotopy fibrations in K-theory. Such a localization theorem was also conjectured by Maltiniotis. Raptis conjectures that μ should not be more than 2-connected in great generality.

One positive result is a theorem of Cisinski and Neeman [CN08] that derivator K-theory of triangulated derivators satisfies an additivity theorem, to be made more explicit shortly. Muro and Raptis in their second paper on derivator K-theory [MR17] asked whether this additivity proof could be adapted to the more general context of left pointed derivators. The positive answer to this question occupies the next section.

5. ADDITIVITY

Rather than approach the problem as Cisinski and Neeman did in [CN08] using Neeman’s method of regions, we will prove additivity in a novel way. Throughout, let \mathbb{D} be a left pointed derivator defined on $\mathbf{Dia} = \mathbf{Dir}_{\mathbf{f}}$. We adapt this first definition from [CN08, Definition 11.7].

Definition 5.1. Let \mathbb{D} be a left pointed derivator. We define the corresponding *cofiber sequence category* for each $K \in \mathbf{Dir}_{\mathbf{f}}$ by $\mathbb{D}_{\text{cof}}(K) \subset \mathbb{D}^K(\square)$ the full subcategory of cocartesian squares X such that $(0, 1)^*X = 0 \in \mathbb{D}(K)$.

Lemma 5.2. The cofiber sequence categories assemble to a prederivator \mathbb{D}_{cof} . Moreover, there is an equivalence $\mathbb{D}^{[1]} \rightarrow \mathbb{D}_{\text{cof}}$ which is pseudonatural with respect to cocontinuous morphisms of derivators, which makes \mathbb{D}_{cof} a left pointed derivator as well.

Proof. For any $u: J \rightarrow K$ in $\mathbf{Dir}_{\mathbf{f}}$, $u^*: \mathbb{D}^{\square}(K) \rightarrow \mathbb{D}^{\square}(J)$ is cocontinuous by [Gro13, Proposition 2.5]. Therefore u^* preserves cocartesian squares and the zero object, so restricts to $u^*: \mathbb{D}_{\text{cof}}(K) \rightarrow \mathbb{D}_{\text{cof}}(J)$. There are no modifications needed for the natural transformations, so this makes \mathbb{D}_{cof} a prederivator.

The equivalence between $\mathbb{D}^{[1]}$ and \mathbb{D}_{cof} is given by the composition of Diagram 3.6:

$$\mathbb{D}^{[1]} \xrightarrow{i_{[1],*}} \mathbb{D}^{\Gamma} \xrightarrow{i_{\Gamma,!}} \mathbb{D}_{\text{cof}} \subset \mathbb{D}^{\square}$$

By definition the image of this composite consists of cocartesian squares with the zero object in the $(0, 1)$ position. Since $i_{[1]}$ and i_{Γ} are fully faithful, their left and right Kan extensions are fully faithful by [Gro13, Proposition 1.20] (which still holds for left pointed derivators), hence the above composite induces an equivalence onto its image, which is precisely $\mathbb{D}_{\text{cof}} \subset \mathbb{D}^{\square}$.

For the pseudonaturality, consider a morphism of derivators $\Phi: \mathbb{D} \rightarrow \mathbb{E}$. Then for $\Phi^{\square}: \mathbb{D}^{\square} \rightarrow \mathbb{E}^{\square}$ to restrict to $\Phi: \mathbb{D}_{\text{cof}} \rightarrow \mathbb{E}_{\text{cof}}$, it would have to send cocartesian squares to cocartesian squares, and it would have to send the zero object of \mathbb{D} to the zero object of \mathbb{E} . As we have assumed Φ is cocontinuous, this property holds and we are done. \square

Remark 5.3. The pseudonaturality with respect to cocontinuous morphisms is the most important takeaway of the preceding lemma. In the below constructions, we will construct morphisms $\Phi: \mathbb{D}_{\text{cof}} \rightarrow (\mathbb{D}_{\text{cof}})^K$ for various categories $K \in \mathbf{Dir}_{\mathbf{f}}$, but often we will have to define these morphisms first as $\Phi: \mathbb{D} \rightarrow \mathbb{D}^K$. We may then extend Φ to a morphism $\mathbb{D}_{\text{cof}} \rightarrow (\mathbb{D}_{\text{cof}})^K$ if Φ is a cocontinuous morphism.

Definition 5.4. Let \mathbb{D}, \mathbb{E} be left pointed derivators. We define a *cofibration morphism of derivators* to be a strict cocontinuous morphism $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$. To Ξ we associate three strict cocontinuous morphisms $\mathbb{D} \rightarrow \mathbb{E}$

$$S := (0, 0)^*\Xi \quad T := (1, 0)^*\Xi \quad Q := (1, 1)^*\Xi$$

and two strict cocontinuous morphisms $\alpha, \beta: \mathbb{D} \rightarrow \mathbb{E}^{[1]}$ given by restricting to the top and right arrows of the coherent square, respectively. Incoherently, we have

$$a \in \mathbb{D} \mapsto \begin{array}{ccc} S(a) & \xrightarrow{\alpha_a} & T(a) \\ \downarrow & & \downarrow \beta_a \\ 0 & \longrightarrow & Q(a) \end{array} \in \mathbb{E}_{\text{cof}}$$

This is a coherent version of a cofibration sequence of exact morphisms of Waldhausen categories in [Wal85, p. 331]. We prove a similar theorem to [Wal85, Proposition 1.3.2].

Theorem 5.5. Let \mathbb{D} and \mathbb{E} be left pointed derivators. The following are equivalent:

- (1) The map

$$\mathbb{D}_{\text{cof}} \xrightarrow{(0,0)^* \times (1,1)^*} \mathbb{D} \times \mathbb{D}$$

induces a homotopy equivalence on derivator K-theory.

- (2) If $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$ is a cofibration morphism of derivators, then there exists a homotopy

$$K(T) \simeq K(S) \sqcup K(Q) (\cong K(S \sqcup Q))$$

The first statement is the statement of additivity à la Garkusha, Maltsiniotis, and Cisinski-Neeman, first found in [Mal07, Conjecture 3] and similar to [Wal85, Proposition 1.3.2(2)]. The latter is a reinterpretation of [Wal85, Proposition 1.3.2(4)]. Our proof follows the strategy set out by Waldhausen.

To expand a little on (1), it will be helpful to use that $\mathbf{S}_{\bullet}\mathbb{D} \times \mathbf{S}_{\bullet}\mathbb{D} \cong \mathbf{S}_{\bullet}\mathbb{D}^{e\sqcup e}$. First, we know that

$$\mathbb{D}(\text{Ar}[n]) \times \mathbb{D}(\text{Ar}[n]) \cong \mathbb{D}(\text{Ar}[n] \sqcup \text{Ar}[n]) \cong \mathbb{D}^{e\sqcup e}(\text{Ar}[n])$$

Second, for a diagram $X \in \mathbb{D}^{e\sqcup e}(\text{Ar}[n])$, the condition of being in $\mathbf{S}_n\mathbb{D}^{e\sqcup e}$ coincides with each projection to $\mathbb{D}(\text{Ar}[n])$ being in $\mathbf{S}_n\mathbb{D}$. This makes it easier for us to define maps into $\mathbf{S}_{\bullet}\mathbb{D} \times \mathbf{S}_{\bullet}\mathbb{D}$; they can arise from (strict) cocontinuous morphisms into $\mathbb{D}^{e\sqcup e}$.

In particular, in the spirit of Remark 5.3, morphisms arising from left adjoint functors are cocontinuous, so for any functor $u: J \rightarrow K$, $u^*, u_!$ induce maps on K-theory. The extension by zero morphisms u_* for any sieve u are also cocontinuous, as they admit an exceptional right adjoint. Put another way, \mathbf{Der}_K contains all left and right Kan extension morphisms available to left pointed derivators.

Proof.

(2) \implies (1)

The map $\rho := (0, 0)^* \times (1, 1)^*$ admits a section $\sigma: \mathbb{D}^{e \sqcup e} \rightarrow \mathbb{D}_{\text{cof}}$ on K-theory. Incoherently for $(a, c) \in \mathbb{D}^{e \sqcup e}$, the functor σ is roughly (but not precisely)

$$(a, c) \mapsto \begin{array}{ccc} a & \rightarrow & a \sqcup c \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c \end{array}$$

We will write σ as a composite of morphisms of derivators coming from diagram functors, but we will need some diagram notation first.

Recall from Notation 3.2 the functor $i_\Gamma: \Gamma \rightarrow \square$. Further, let $i: e \sqcup e \rightarrow \Gamma$ be the inclusion into $(1, 0)$ and $(0, 1)$. We also consider the category $[1] \times [2]$, with labelling

$$\begin{array}{ccc} (0, 0) & \rightarrow & (1, 0) \\ \downarrow & & \downarrow \\ (0, 1) & \rightarrow & (1, 1) \\ \downarrow & & \downarrow \\ (0, 2) & \rightarrow & (1, 2) \end{array}$$

Let J be the full subcategory of $[1] \times [2]$ without the element $(1, 2)$. Finally, let $i_\square: \square \rightarrow J$ and $j: J \rightarrow [1] \times [2]$ be the obvious inclusions, and let $r: \square \rightarrow [1] \times [2]$ be the inclusion of the bottom square. Note that i is a cosieve, and i_Γ, i_\square , and j are sieves.

At the level of the derivators, the section σ is given by

$$(5.6) \quad \mathbb{D}^{e \sqcup e} \xrightarrow{i_!} \mathbb{D}^\Gamma \xrightarrow{i_\Gamma^!} \mathbb{D}^\square \xrightarrow{i_\square^*} \mathbb{D}^J \xrightarrow{j_!} \mathbb{D}^{[1] \times [2]} \xrightarrow{r^*} \mathbb{D}^\square \supset \mathbb{D}_{\text{cof}}$$

All of these maps are cocontinuous, though not all are strict. As we mentioned in Corollary 4.2, this means that σ will give rise to a well-defined map in \mathcal{S} .

For $(a, c) \in \mathbb{D}^{e \sqcup e}$, $\sigma(a, c)$ is explicitly

$$(a, c) \mapsto \begin{array}{ccc} 0 & \rightarrow & c \\ \downarrow & & \downarrow \\ a & & \end{array} \mapsto \begin{array}{ccc} 0 & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \rightarrow & a \sqcup c \end{array} \mapsto \begin{array}{ccc} 0 & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \rightarrow & a \sqcup c \\ \downarrow & & \downarrow \\ 0 & & \end{array} \mapsto \begin{array}{ccc} 0 & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \rightarrow & a \sqcup c \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c' \end{array} \mapsto \begin{array}{ccc} a & \rightarrow & a \sqcup c \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c' \end{array}$$

By construction, the image of this composite lands in $\mathbb{D}_{\text{cof}} \subset \mathbb{D}^{\square}$, as the square $r^*j_!X$ is easily shown to be cocartesian using Proposition 6.3 for any $X \in \mathbb{D}^J$, and $(1, 0)^*r^*j_!X = 0$ as long as X is in the image of $i_{\square,*}$ (as is our case). Note further that the composite map $c \rightarrow a \sqcup c \rightarrow c'$ is an isomorphism, as it is the pushout of the isomorphism $0 \rightarrow 0$.

All this data together gives us the section $\sigma: \mathbb{D}^{e \sqcup e} \rightarrow \mathbb{D}_{\text{cof}}$. Hence we obtain an isomodification $\text{id}_{\mathbb{D}^{e \sqcup e}} \Rightarrow \rho\sigma$, as the canonical isomorphism $c \rightarrow c'$ gives rise to an isomorphism $(a, c) \rightarrow \rho\sigma(a, c) = (a, c')$ natural in $(a, c) \in \mathbb{D}^{e \sqcup e}$. On K-theory (after strictifying the non-strict morphisms and passing to \mathcal{S}), this gives a homotopy $K(\rho\sigma) \simeq K(\text{id}_{\mathbb{D}^{e \sqcup e}})$. Therefore it suffices to construct a homotopy in the reverse direction, i.e. $K(\sigma\rho) \simeq K(\text{id}_{\mathbb{D}_{\text{cof}}})$.

To that end, we use our assumption. We will construct a cofibration morphism of derivators such that $S \sqcup Q \cong \sigma\rho$ and $T \cong \text{id}_{\mathbb{D}_{\text{cof}}}$. Our morphism $\Xi: \mathbb{D}_{\text{cof}} \rightarrow (\mathbb{D}_{\text{cof}})_{\text{cof}}$ will have the form

$$(5.7) \quad \begin{array}{ccc} & & \begin{array}{ccc} a \xrightarrow{=} a & & a \xrightarrow{f} b \\ \downarrow & \downarrow & \downarrow \text{id}_a, f \\ 0 \rightarrow 0 & \longrightarrow & 0 \rightarrow c \end{array} \\ & & \downarrow \\ & & \begin{array}{ccc} a \xrightarrow{f} b & & a \xrightarrow{f} b \\ \downarrow & \downarrow & \downarrow g \\ 0 \rightarrow c & \longrightarrow & 0 \rightarrow c \end{array} \\ & & \downarrow g, \text{id}_c \\ & & \begin{array}{ccc} 0 \rightarrow 0 & & 0 \rightarrow c \\ \downarrow & \downarrow & \downarrow \\ 0 \rightarrow 0 & \longrightarrow & 0 \rightarrow c \end{array} \\ & & \downarrow = \end{array}$$

where all squares commute.

We can accomplish this as the pullback along a single functor in $\mathbf{Dir}_{\mathbf{f}}$. In the diagram above of shape $\square \times \square$, let the first \square denote the outer square coordinates and the second the inner coordinates. For example, the entry c in the top right is at $(1, 0, 1, 1)$. We now define $\xi: \square \times \square \rightarrow \square$ by

$$\xi(a_1, b_1, a_2, b_2) = \begin{cases} (0, 0) & (a_1, b_1, a_2, b_2) = (0, 0, 0, 0), (0, 0, 1, 0), (1, 0, 0, 0) \\ (1, 0) & (a_1, b_1, a_2, b_2) = (1, 0, 1, 0) \\ (1, 1) & (a_1, b_2, a_2, b_2) = (1, 0, 1, 1), (1, 1, 1, 0), (1, 1, 1, 1) \\ (0, 1) & \text{otherwise} \end{cases}$$

In plain language, we make this definition so that ξ behaves on objects as sketched in Diagram 5.7, and it is also the appropriate functor for the maps. For example, consider the map $(1, 0, 0, 0) \rightarrow (1, 0, 1, 0)$ in $\square \times \square$. Diagram 5.7 says that we want

$$(1, 0, 0, 0)^*\xi^*X \rightarrow (1, 0, 1, 0)^*\xi^*X = a \xrightarrow{f} b$$

We see that $\xi(1, 0, 0, 0) = (0, 0)$ and $\xi(1, 0, 1, 0) = (1, 0)$, and thus the natural transformation $(\xi(1, 0, 0, 0))^* \Rightarrow (\xi(1, 0, 1, 0))^*$ induces the map $f: a \rightarrow b$ when applied to $X \in \mathbb{D}_{\text{cof}}$. Checking that we also have g and identities where required can be done similarly.

We define a strict cocontinuous morphism of derivators $\Xi := \xi^*: \mathbb{D}^{\square} \rightarrow \mathbb{D}^{\square \times \square}$. By construction, assuming we restrict our domain to \mathbb{D}_{cof} (as illustrated), the image will be a global cocartesian square in $\mathbb{D}_{\text{cof}}(\square)$ with zero in the bottom-left corner, so we obtain the required (strict cocontinuous) morphism $\Xi: \mathbb{D}_{\text{cof}} \rightarrow (\mathbb{D}_{\text{cof}})_{\text{cof}}$.

Our assumption gives us that $K(T) \simeq K(S \sqcup Q)$, and clearly $T = \text{id}_{\mathbb{D}_{\text{cof}}}$. It is also evident that we have an isomodification $\sigma\rho \Rightarrow S \sqcup Q$ using again the naturality of the comparison isomorphism $c' \rightarrow c$. This shows that $K(\text{id}_{\mathbb{D}_{\text{cof}}}) \simeq K(\sigma\rho)$ as required, which proves additivity in the historical sense for derivator K-theory.

(1) \implies (2)

First, consider the two maps $(1, 0)^*: \mathbb{E}_{\text{cof}} \rightarrow \mathbb{E}$ and $\bar{\rho}: \mathbb{E}_{\text{cof}} \rightarrow \mathbb{E}^{e\sqcup e} \rightarrow \mathbb{E}$, where $\bar{\rho}$ is defined to be the composite cocontinuous morphism $(1, 1)^* i_{\Gamma} i_{\Delta} \rho$ which computes the coproduct of $(0, 0)^* X$ and $(1, 1)^* X$ for any $X \in \mathbb{E}_{\text{cof}}$ (see Equation 5.6).

We claim that these maps are homotopic. If we precompose with the map $\sigma: \mathbb{E}^{e\sqcup e} \rightarrow \mathbb{E}_{\text{cof}}$, it is immediate that $(1, 0)^* \sigma$ and $\bar{\rho} \sigma$ are (canonically) isomorphic, as they both compute the coproduct of $(a, c) \in \mathbb{E}^{e\sqcup e}$. Thus if σ is a homotopy equivalence on K-theory, $(1, 0)^*$ and $\bar{\rho}$ are still homotopic. But by our assumption (1), σ is a section of the homotopy equivalence ρ , so it too is a homotopy equivalence.

Statement (2) then follows immediately by precomposing these two homotopic maps by any cofibration morphism $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$, which yields

$$K(\bar{\rho}\Xi) \cong K(S \sqcup Q) \simeq K(T) = K((1, 0)^*\Xi)$$

□

This new reformulation of the additivity theorem produces a proof that differs greatly from [CN08] and [Gar05], the latter of which includes a gap which seems irreparable, see [MR17, §6.3]. We will now prove Theorem 5.5(2) in the remainder of this section, following a strategy given by Grayson in [Gra11]. Grayson's paper proves the (classical) additivity theorem for Waldhausen K-theory using an explicit combinatorial homotopy. We are able to take this diagram-flavored argument and make it homotopy-coherent enough for application to derivator K-theory.

Let Y be a simplicial set. Without any loss of generality, we may extend $Y: \Delta^{\text{op}} \rightarrow \mathbf{Sets}$ to $Y: \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Sets}$, where \mathbf{Ord} is the category of (nonempty) finite totally ordered sets with order-preserving maps. For $A \in \mathbf{Ord}$, we let $Y(A) := Y([n])$, where $[n]$ is the unique element of $\Delta \subset \mathbf{Ord}$ isomorphic to A . We do this in order to introduce a binary operation $*$ on \mathbf{Ord} , which otherwise would

cause us problems. We let $A * B$ be concatenation, that is,

$$A * B := (\{0\} \times A) \cup (\{1\} \times B) \subset [1] \times (A \cup B)$$

with lexicographical ordering. This results in each element of A being smaller than each element of B , but within A and B the ordering does not change.

Definition 5.8. Let Y be a simplicial set (on **Ord**). The *two-fold edge-wise subdivision* $\text{sub}_2 Y$ of Y is the simplicial set defined by $\text{sub}_2 Y(A) := Y(A * A)$.

There is a natural homeomorphism $|\text{sub}_2 Y| \rightarrow |Y|$ defined in [Gra89, §4] whose construction we do not recall here. The important thing to note is that we do not change the homotopy type (or even homeomorphism type) of our simplicial set by subdividing.

Now we can begin to bring derivators back into the conversation. Let $\Phi, \Psi: \mathbb{D} \rightarrow \mathbb{E}$ be two strict cocontinuous morphisms of derivators. These induce morphisms of simplicial categories $\mathbf{S}_\bullet \Phi, \mathbf{S}_\bullet \Psi: \mathbf{S}_\bullet \mathbb{D} \rightarrow \mathbf{S}_\bullet \mathbb{E}$. We define a new map of simplicial categories $\nabla_{\Phi, \Psi}: \text{sub}_2 \mathbf{S}_\bullet \mathbb{D} \rightarrow \mathbf{S}_\bullet \mathbb{E}$ in the following way.

The totally ordered set $A * A$ has two full subcategories $i_0, i_1: A \rightarrow A * A$, given by $a \mapsto (0, a), (1, a)$ respectively. These extend to functors on the arrow categories $\text{Ar}(A) \rightarrow \text{Ar}(A * A)$, and give $i_0^*, i_1^*: \mathbb{D}(\text{Ar}(A * A)) \rightarrow \mathbb{D}(\text{Ar}(A))$. Since restriction morphisms are strict and cocontinuous, i_0^* and i_1^* define morphisms of simplicial categories $\mathbf{S}_\bullet \mathbb{D}(A * A) \rightarrow \mathbf{S}_\bullet \mathbb{D}(A)$, where we adopt the notation $\mathbf{S}_\bullet \mathbb{D}(A) = \mathbf{S}_n \mathbb{D}$ for the unique $[n]$ such that $A \cong [n]$.

Consider an object $X \in \text{sub}_2 \mathbf{S}_\bullet \mathbb{D}(A) = \mathbf{S}_\bullet \mathbb{D}(A * A)$. Then let

$$\nabla_{\Phi, \Psi}(X) := \Phi(i_0^*(X)) \sqcup \Psi(i_1^*(X))$$

This indeed lands in $\mathbf{S}_\bullet \mathbb{E}(A)$ as the coproduct of any two cocartesian squares is again cocartesian. We have, in essence, doubled $\mathbf{S}_\bullet \mathbb{D}$ and applied Φ to the first half and Ψ to the second, then taken the coproduct of the results.

Unfortunately, the described map does not exist on the level of simplicial categories. While the functors Φi_0^* and Ψi_1^* are still strict, taking the coproduct is not a strict operation. Therefore we will need to strictify this last map, as in Proposition 4.1, so we do not honestly get a map of simplicial sets with codomain $\mathbf{S}_\bullet \mathbb{E}$. But on K-theory the map we want does exist, given by the zig-zag

(5.9)

$$\Omega | \text{sub}_2 N_\bullet i \mathbf{S}_\bullet \mathbb{D} | \xrightarrow{K(\Phi i_0^*) \times K(\Psi i_1^*)} K(\mathbb{E}) \times K(\mathbb{E}) \cong K(\mathbb{E}^{e \sqcup e}) \xleftarrow{\sim} K(\widetilde{\mathbb{E}^{e \sqcup e}}) \xrightarrow{\sqcup} K(\mathbb{E})$$

where $\widetilde{\mathbb{E}^{e \sqcup e}}$ is the prederivator constructed in Proposition 4.1 to strictify the coproduct map $\sqcup = (1, 1)^* i_{\Gamma, !} i_{!} : \mathbb{E}^{e \sqcup e} \rightarrow \mathbb{E}$.

Now, we need to construct a cylinder object for $\mathbf{S}_\bullet \mathbb{D}$ that will allow us to use $\nabla_{\Phi, \Psi}$ as a replacement for $\Phi \sqcup \Psi$. We need three definitions to get us there.

Definition 5.10 (Definition 1.2, [Gra11]). For $A, B \in \mathbf{Ord}$, let $A \times B$ be the set $A \times B$ with lexicographic ordering. That is, $(a, b) \leq (a', b')$ if and only if $a < a'$ or $a = a'$ and $b \leq b'$. Note that the map $A \times B \rightarrow A$ is order-preserving, hence a morphism in \mathbf{Ord} , but the other ‘projection’ $A \times B \rightarrow B$ is generally not.

Definition 5.11 (Definition 1.3, [Gra11]). Given two maps $\varphi: A \rightarrow C$ and $s: B \rightarrow C$ in \mathbf{Ord} , define $\varphi^{-1}(s) \in \mathbf{Ord}$ to be the subset of $A \times B$ given by $\{(a, b) : \varphi(a) = s(b)\}$.

Definition 5.12 (Definition 1.4, [Gra11]). Let $s: [2] \rightarrow [1]$ be the morphism defined by $s(0) = 0$ and $s(1) = s(2) = 1$. For any simplicial set Y , define a new simplicial set IY by $IY(A) := \{(\varphi, y) : \varphi: A \rightarrow [1], y \in Y(\varphi^{-1}(s))\}$. The definition of IY on morphisms in \mathbf{Ord} extends by naturality.

Remark 5.13. To see how IY is a useful object, notice that

$$\varphi^{-1}(s) = \varphi^{-1}(0) * \varphi^{-1}(1) * \varphi^{-1}(1)$$

Therefore the choice $\varphi = 0$ gives $\varphi^{-1}(s) = A$, and the choice $\varphi = 1$ gives $\varphi^{-1}(s) = A * A$. The simplicial subset of IY at $\varphi = 0$ is isomorphic to Y , and the simplicial subset of IY at $\varphi = 1$ is isomorphic to $\text{sub}_2 Y$. Any other morphism $\varphi: A \rightarrow [1]$ gives a totally ordered set $\varphi^{-1}(s)$ interpolating between these two endpoints.

Lemma 5.14 (Lemma 1.6, [Gra11]). There is a homeomorphism $|IY| \rightarrow |\Delta^1| \times |Y|$.

We do not include the proof because nothing is changed in the context of derivators. This shows that IY is indeed a cylinder object for Y , so we may prove the main proposition.

Proposition 5.15. Let $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$ be a cofibration morphism of derivators, and let $S, T, Q: \mathbb{D} \rightarrow \mathbb{E}$ be the corresponding morphisms of derivators. Then there is a map of simplicial categories $\Theta: \mathbf{IS}_\bullet \mathbb{D} \rightarrow \mathbf{S}_\bullet \mathbb{E}$ such that Θ agrees with T on the simplicial subcategory where $\varphi = 0$ and Θ agrees with $\nabla_{Q,S}$ on the simplicial subcategory where $\varphi = 1$.

Remark 5.16. Just as $\nabla_{Q,S}$ is not an honest morphism of simplicial categories, Θ will not be well-defined per se but will induce a map on K-theory via a zigzag coming from strictification. We will continue to abuse notation in this fashion.

Proof. We will construct Θ in two steps.

First, we define a morphism $P: \mathbb{D}^{[1]} \rightarrow \mathbb{E}$. For a coherent morphism $(f: a \rightarrow b) \in \mathbb{D}^{[1]}$, we may apply $\Xi^{[1]}$ to obtain an object in $\mathbb{E}_{\text{cof}}^{[1]} \subset \mathbb{E}^{[1] \times \square}$. Specifically,

its underlying diagram takes the form (where we do not label every arrow)

$$\begin{array}{ccccc}
 S(a) & \xrightarrow{\alpha_a} & T(a) & & \\
 \downarrow s(f) & \searrow & \downarrow & \searrow \beta_a & \\
 0 & \longrightarrow & Q(a) & & \\
 \downarrow & & \downarrow & & \downarrow Q(f) \\
 S(b) & \longrightarrow & T(b) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \longrightarrow & Q(b) & &
 \end{array}$$

We may consider the functor $\Gamma \rightarrow [1] \times \square$ by the inclusion into the upper-left corner of the back face of the cube given above. Restriction along this functor gives the coherent diagram in \mathbb{E}^Γ

$$(5.17) \quad \begin{array}{c}
 S(a) \xrightarrow{\alpha_a} T(a) \\
 \downarrow s(f) \\
 S(b)
 \end{array}$$

We may then apply $(1,1)^*i_{\Gamma,!}$ to first take the pushout of Diagram 5.17 and then restrict to the new object. The composition is a cocontinuous morphism which we denote $P: \mathbb{D}^{[1]} \rightarrow \mathbb{E}$.

We point out two special cases. If $f \in \mathbb{D}^{[1]}$ is a coherent isomorphism, then $P(f) \cong T(a)$, as pushing out along an isomorphism is still an isomorphism. Second, if f is a zero map, then there is a natural isomorphism $P(f) \cong Q(a) \sqcup S(b)$. This arises from a factorization of f as follows

$$\begin{array}{c}
 S(a) \rightarrow T(a) \\
 \downarrow \\
 0 \\
 \downarrow \\
 S(b)
 \end{array}
 \begin{array}{c}
 \curvearrowright \\
 s(f) \\
 \curvearrowleft
 \end{array}$$

If we take pushouts one square at a time, we first obtain an object isomorphic to $Q(a)$ and second compute the pushout of $Q(a)$ with $S(b)$:

$$\begin{array}{ccc}
 \begin{array}{c} S(a) \rightarrow T(a) \\ \downarrow \\ 0 \\ \downarrow \\ S(b) \end{array} & \mapsto & \begin{array}{c} S(a) \rightarrow T(a) \\ \downarrow \quad \downarrow \\ 0 \rightarrow Q(a) \\ \downarrow \\ S(b) \end{array} & \mapsto & \begin{array}{c} S(a) \longrightarrow T(a) \\ \downarrow \quad \downarrow \\ 0 \longrightarrow Q(a) \\ \downarrow \quad \downarrow \\ S(b) \rightarrow S(b) \sqcup Q(a) \end{array}
 \end{array}$$

But the composition of two pushouts is again a pushout, which implies that $S(b) \sqcup Q(a)$ is the pushout of the total upper-right corner, which is exactly Diagram 5.17. Thus we conclude $P(f) \cong S(b) \sqcup Q(a)$.

Remark 5.18. The phrase ‘coherent isomorphism’ is justified in any (not strong) left derivator \mathbb{D} : an object $f \in \mathbb{D}([1])$ has underlying diagram an isomorphism if and only if the cone $C(f) \cong 0 \in \mathbb{D}(e)$. As another option, $f \in \mathbb{D}([1])$ is a coherent isomorphism if and only if the counit of the $(0_!, 0^*)$ adjunction is an isomorphism by [Gro13, Proposition 3.12]. This is actually true in any prederivator, as the functor $0_! : \mathbb{D}(e) \rightarrow \mathbb{D}([1])$ is canonically isomorphic to $\pi_{[1]}^*$, where $\pi_{[1]} : [1] \rightarrow e$ is the projection.

Unfortunately, we do not have a way to make sense of a ‘coherent zero map’, i.e. an object in $\mathbb{D}([1])$ whose underlying diagram factors through the 0 in $\mathbb{D}(e)$. Just as there is no morphism $\mathbb{D}([1]) \sqcup_{\mathbb{D}(e)} \mathbb{D}([1]) \rightarrow \mathbb{D}([2])$ that would ‘coherently compose’ two coherent maps with the appropriate domain and codomain, there is no morphism $\mathbb{D}([1]) \rightarrow \mathbb{D}([2])$ which ‘inserts’ the zero object.

Instead, what we do is take the underlying diagram of Diagram 5.17, and notice that the vertical map factors through zero. We may then lift the ‘tall Γ ’ to a coherent object as this category is still finite free. The pushouts maybe taken one step at a time, similar to the construction at Equation 5.6, to give us a coherent object of $\mathbb{D}([1] \times [2])$, whose restriction to the outer square gives us the original pushout of Diagram 5.17.

The step of passing to the incoherent diagram, inserting zero, and lifting back to a slightly-larger coherent diagram is not functorial, but it does allow us to identify (up to isomorphism) the pushout of our original diagram. It would be helpful to know if there is an appropriate notion of ‘coherent zero map’, in which case we could eliminate the assumption of strongness from our left pointed derivators. It would be equally interesting to know if the notion of ‘coherent zero map’ does not exist for non-strong derivators, thereby validating all the axioms on the domain of derivator K-theory. We leave this for future work.

Now, we construct the second step of Θ . Recall the definition of $s : [2] \rightarrow [1]$. Now we define two sections of s : first, $d : [1] \rightarrow [2]$ with $d(0) = 0$, $d(1) = 1$; second, $e : [1] \rightarrow [2]$ with $e(0) = 0$, $e(1) = 2$. For any $a \in A$, $(d\varphi(a), a)$ and $(e\varphi(a), a)$ are both elements of $\varphi^{-1}(s)$ as $se = sd = \text{id}_{[1]}$. This gives two inclusions from A into $\varphi^{-1}(s)$, to which we also name d and e (as the other functors will not be returning). There is also a unique natural transformation $\zeta : d \Rightarrow e$ as $d\varphi(a) \leq e\varphi(a)$ for any $a \in A$.

We may first extend $d, e : \text{Ar}(A) \rightarrow \text{Ar}(\varphi^{-1}(s))$ by naturality, and we still have the natural transformation $\zeta : d \Rightarrow e$. We can better view ζ as a functor $\zeta : \text{Ar}(A) \times [1] \rightarrow \text{Ar}(\varphi^{-1}(s))$, where (in particular) the original data of

the natural transformation is retained by the maps in the $[1]$ -dimension

$$\zeta\left(\left((a \rightarrow b), 0\right) \rightarrow \left((a \rightarrow b), 1\right)\right) = d(a \rightarrow b) \xrightarrow{\zeta_{(a \rightarrow b)}} e(a \rightarrow b).$$

Therefore we obtain a strict cocontinuous morphism

$$\zeta^*: \mathbb{D}(\text{Ar}(\varphi^{-1}(s))) \rightarrow \mathbb{D}(\text{Ar}(A) \times [1])$$

As we vary $\varphi: A \rightarrow [1]$, we obtain a map out of $IS_{\bullet}\mathbb{D}(A)$, and by cocontinuity we can restrict the codomain to $\mathbf{S}_{\bullet}\mathbb{D}^{[1]}(A)$. This tells us how to extend the various ζ^* to a map $Z: IS_{\bullet}\mathbb{D} \rightarrow \mathbf{S}_{\bullet}\mathbb{D}^{[1]}$ (read: capital ζ).

We now need to check that Z is a map of simplicial categories. Suppose we have a map $\psi: B \rightarrow A$ in \mathbf{Ord} . Then we need to check that the following square commutes:

$$(5.19) \quad \begin{array}{ccc} IS_{\bullet}\mathbb{D}(A) & \xrightarrow{\psi_1^*} & IS_{\bullet}\mathbb{D}(B) \\ Z_A \downarrow & & \downarrow Z_B \\ \mathbf{S}_{\bullet}\mathbb{D}^{[1]}(A) & \xrightarrow{\psi_2^*} & \mathbf{S}_{\bullet}\mathbb{D}^{[1]}(B) \end{array}$$

where we use ψ_1^*, ψ_2^* to denote the maps induced by ψ on the two different simplicial categories $IS_{\bullet}\mathbb{D}$ and $\mathbf{S}_{\bullet}\mathbb{D}^{[1]}$.

Let $(\varphi: A \rightarrow [1], X \in \mathbf{S}_{\bullet}\mathbb{D}(\varphi^{-1}(s))) \in IS_{\bullet}\mathbb{D}(A)$. Then taking the upper composition of Diagram 5.19 we first get the object

$$\psi_1^*(\varphi, X) = (\varphi\psi, (\psi')^*X \in \mathbf{S}_{\bullet}\mathbb{D}((\varphi\psi)^{-1}(s)))$$

where ψ' is the natural map $(\varphi\psi)^{-1}(s) \rightarrow \varphi^{-1}(s)$ induced by ψ , $(n, b) \mapsto (n, \psi(b))$. If we apply Z_B to $(\varphi\psi, (\psi')^*X)$, we obtain the coherent object in $\mathbf{S}_{\bullet}\mathbb{D}^{[1]}(B)$

$$d_B^*(\psi')^*X \xrightarrow{\zeta_B^*} e_B^*(\psi')^*X$$

where d_B and e_B are the specific instances of d, e in this case.

If we traverse Diagram 5.19 using the lower composition, we first take $Z_A(\varphi, X)$ to obtain $\zeta_A^*: d_A^*X \rightarrow e_A^*X$. Then applying ψ_2^* we obtain

$$\psi^*d_A^*X \xrightarrow{\psi^*\zeta_A^*} \psi^*e_A^*X.$$

But by strict 2-functoriality, these two coherent maps are equal on the nose, as they are just pullbacks induced by maps in \mathbf{Dir}_f . Therefore Z is indeed a morphism of simplicial categories.

This finishes the definition of $\Theta = (\mathbf{S}_{\bullet}P)Z: IS_{\bullet}\mathbb{D} \rightarrow \mathbf{S}_{\bullet}\mathbb{D}^{[1]} \rightarrow \mathbf{S}_{\bullet}\mathbb{E}$. We now need to check that Θ in fact interpolates between T and $\nabla_{Q,s}$. Suppose we take some $A \in \mathbf{Ord}$, and let $(\varphi, X) \in IS_{\bullet}\mathbb{D}(A)$.

If $\varphi = 0$ is the zero map, then $\varphi^{-1}(s) = A$ by Remark 5.13. In this case, $\zeta: d \Rightarrow e$ is the identity natural transformation, as $d\varphi(a) = e\varphi(a)$ always. So for an element $(0, X \in \mathbf{S}_\bullet \mathbb{D}(A)) \in \mathbf{IS}_\bullet \mathbb{D}(A)$, we see that $Z_A(0, X)$ is nothing else but the constant map in the $[1]$ -direction. Thus $Z_A(0, X) = \text{id}_X$. This is an isomorphism, one of the special cases we noted following Diagram 5.17, and we conclude that $\mathbf{S}_\bullet P(Z_A(0, X))$ is naturally isomorphic to $\mathbf{S}_\bullet T(X)$.

If $\varphi = 1$, then $\varphi^{-1}(s) = A * A$ by the same remark. We now consider an object of the form $(1, X \in \mathbf{S}_\bullet \mathbb{D}(A * A)) \in \mathbf{IS}_\bullet \mathbb{D}(A)$. The resulting $Z_A(1, X)$ is an element of $\mathbf{S}_\bullet \mathbb{D}^{[1]}(A)$, which is a subcategory of $\mathbb{D}(\text{Ar}(A) \times [1])$. Incoherently, if we write $X = (x \rightarrow y)$, as $X \in \mathbb{D}(\text{Ar}(A * A))$, then $Z_A(1, X)$ looks like

$$\begin{array}{ccc} d^*x & \xrightarrow{\zeta_x^*} & e^*x \\ \downarrow & & \downarrow \\ d^*y & \xrightarrow{\zeta_y^*} & e^*y \end{array}$$

where the vertical maps are the elements of $\text{Ar}(A * A)$. But since $\varphi = 1$, we know that $e^*x \geq d^*y$. Therefore the map above factors as

$$\begin{array}{ccccc} d^*x & \longrightarrow & d^*y & \longrightarrow & e^*x \\ \downarrow & & \downarrow & & \downarrow \\ d^*y & \longrightarrow & d^*y & \longrightarrow & e^*y \end{array}$$

Therefore our overall (coherent) map $Z_A(1, X)$ factors through the object $(d^*y \rightarrow d^*y) \in \mathbf{S}_\bullet \mathbb{D}(A)$. But this is a zero object by assumption, as it corresponds to an object of the form $(i, i) \in \text{Ar}([n])$ (where $[n] \cong A * A$), so $Z_A(1, X)$ is a zero map. This was our second special case, and we conclude that $\mathbf{S}_\bullet P(Z_A(1, X))$ is naturally isomorphic to $\mathbf{S}_\bullet \nabla_{Q,S}(X)$. This completes the proof. \square

Proving the additivity of derivator K-theory is now immediate.

Theorem 5.20. Let $\Xi: \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$ be a cofibration morphism of derivators, and let S, T, Q be the corresponding morphisms of derivators. Then $S \sqcup Q$ and T induce homotopic maps $K(\mathbb{D}) \rightarrow K(\mathbb{E})$.

Proof. From Lemma 5.14 and Proposition 5.15, the morphism of simplicial categories Θ above gives a homotopy between $\mathbf{S}_\bullet T$ and $\nabla_{\mathbf{S}_\bullet Q, \mathbf{S}_\bullet S}$ on $|\mathbf{S}_\bullet \mathbb{D}| \rightarrow |\mathbf{S}_\bullet \mathbb{E}|$. We now need to prove that there exists a cofibration morphism of derivators whose associated functors are $S, S \sqcup Q, Q$.

From the proof of Proposition 5.5, recall that we constructed a functor $\sigma: \mathbb{E}^{e \sqcup e} \rightarrow \mathbb{E}_{\text{cof}}$ whose image is precisely what we desire. Let us precompose σ by the morphism $(S \sqcup Q): \mathbb{D} \rightarrow \mathbb{E}^{e \sqcup e}$. The target map of $\sigma(S \sqcup Q): \mathbb{D} \rightarrow \mathbb{E}_{\text{cof}}$

is isomorphic to $S \sqcup Q$, and so $\nabla_{\mathbf{S}\bullet Q, \mathbf{S}\bullet S}$ and $\mathbf{S}\bullet S \sqcup \mathbf{S}\bullet Q$ induce homotopic maps on K-theory as well. Combining these homotopies, we see that T and $S \sqcup Q$ induce homotopic maps $K(\mathbb{D}) \rightarrow K(\mathbb{E})$. \square

In particular, by Proposition 5.5, this proves additivity as it was conjectured by Maltsiniotis in [Mal07].

6. FURTHER PROPERTIES

As additivity sits as the central property of any flavor of algebraic K-theory, we can now see what else we have obtained. First, as promised, we can perform the Eilenberg swindle for left pointed derivators with ‘large’ domains.

Proposition 6.1. Suppose that \mathbb{D} is a left pointed derivator defined on $\mathbf{Dia} \subset \mathbf{Cat}$ such that the countable discrete set ω is in \mathbf{Dia} . Then $K(\mathbb{D}) \cong 0$.

Proof. If we look at the functor $\pi_\omega: \omega \rightarrow e$, we see that the left Kan extension $\pi_{\omega,!}$ is the colimit of this discrete set, i.e. the coproduct. From an object $X \in \mathbb{D}(e)$, we can obtain the countable coproduct of copies of X via the functor $\pi_{\omega,!}\pi_\omega^*: \mathbb{D}(e) \rightarrow \mathbb{D}(e)$, and we can promote this to a cocontinuous morphism of derivators $\coprod: \mathbb{D} \rightarrow \mathbb{D}$.

We now want to construct a cofibration morphism of derivators with this construction in mind. Let $s: \omega \rightarrow \omega$ denote the successor function, i.e. $s(n) = n + 1$. We build another category Γ_s as the diagram underlying the function s . That is, the objects of Γ_s are two disjoint copies of ω , which we label 0 and 1, with an arrow $n_0 \rightarrow s(n)_1 = (n + 1)_1$. To sketch the initial segment of Γ_s :

$$\begin{array}{ccccccccc} 0_0 & & 1_0 & & 2_0 & & 3_0 & & \cdots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ 0_1 & & 1_1 & & 2_1 & & 3_1 & & 4_1 & & \cdots \end{array}$$

This category admits a functor $p: \Gamma_s \rightarrow [1]$ which sends n_i to $i \in [1]$. We claim that the cofibration morphism of derivators $\Xi: \mathbb{D} \rightarrow \mathbb{D}_{\text{cof}}$ induced by

$$\mathbb{D} \xrightarrow{\pi_{\Gamma_s}^*} \mathbb{D}^{\Gamma_s} \xrightarrow{p!} \mathbb{D}[1] \xrightarrow{i_{[1],*}} \mathbb{D}^\Gamma \xrightarrow{i_{\Gamma,!}} \mathbb{D}_{\text{cof}}$$

provides us with a null homotopy of the identity on \mathbb{D} , giving us our swindle.

We need to identify the source, target, and quotient morphisms associated to this cofibration morphism. Let $X \in \mathbb{D}$. To begin, $(0, 0)^*\Xi$ is isomorphic to $0^*p!\pi_{\Gamma_s}^*$, as the last two functors $i_{[1],*}$ and $i_{\Gamma,!}$ are fully faithful and do not change the underlying diagram of $[1] \subset \square$. Using Der4L, we can compute $0^*p!\pi_{\Gamma_s}^* X$ via the comma category $(p/0)$. Specifically,

$$0^*p!\pi_{\Gamma_s}^* X \cong \pi_{(p/0),!}\text{Pr}^*\pi_{\Gamma_s}^* X$$

where $\text{pr}: (p/0) \rightarrow \Gamma_s$ is the usual projection to the comma category as in Definition 2.4. The composition $\text{pr}^* \pi_{\Gamma_s}^*$ is the pullback along $(p/0) \rightarrow \Gamma_s \rightarrow e$, which is the same thing as the pullback $\pi_{(p/0)}^*$ directly. To complete this computation, we need to identify the comma category.

The objects of the comma category are $n_i \in \Gamma_s$ along with a map $p(n_i) = i \rightarrow 0$. This forces $p(n_i) = 0$ as there are no other maps in $[1]$. Therefore $(p/0)$ consists solely of the subcategory $\omega_0 \subset \Gamma_s$. Therefore we conclude

$$(0, 0)^* \Xi X \cong 0^* p_! \pi_{\Gamma_s}^* X \cong \pi_{(p/0), !} \text{pr}^* \pi_{\Gamma_s}^* X \cong \pi_{\omega_0, !} \pi_{\omega_0}^* X = \coprod X$$

The computation for $(1, 0)^* \Xi X$ is similar: it is isomorphic to $1^* p_! \pi_{\Gamma_s}^* X$. We have an isomorphism by Der4L

$$1^* p_! \pi_{\Gamma_s}^* X \cong \pi_{(p/1), !} \text{pr}^* \pi_{\Gamma_s}^* X \cong \pi_{(p/1), !} \pi_{(p/1)}^* X$$

We now need to identify $(p/1)$ as a category. As $1 \in [1]$ is the terminal object, this comma category is equal to Γ_s itself. In order to compute the colimit of shape Γ_s , we note that it admits a reflective subcategory. Consider $\omega_1 \subset \Gamma_s$ as a subcategory. Then this inclusion admits a left adjoint $\ell: \Gamma_s \rightarrow \omega_1$ such that $\ell(n_1) = n_1$ and $\ell(m_0) = S(m_0) = (m+1)_1$. To double check that this is actually an adjunction, we check for $m_0, n_1 \in \Gamma_s$

$$\text{Hom}_{\Gamma_s}(m_0, n_1) = \text{Hom}_{\omega_1}((m+1)_1, n_1)$$

The lefthand side is nonempty if and only if $m+1 = n$, which is the same for the righthand side. In the case that we are looking at the maps between m_0 and n_0 or m_1 and n_2 , the hom-set equality is straightforward as both sides are empty.

By [Gro13, Proposition 1.18], right adjoint functors preserve colimits; that is, if $r: \omega_1 \rightarrow \Gamma_s$ is the inclusion, we have $\pi_{\omega_1, !} r^* Y \cong \pi_{\Gamma_s, !} Y$ for any $Y \in \mathbb{D}(\Gamma_s)$. Noting finally that the composition $r^* \pi_{\omega_1}^* = \pi_{(p/1)}^*$, we can now complete the chain of isomorphisms to finish the computation:

$$(1, 0)^* \Xi X \cong 1^* p_! \pi_{\Gamma_s}^* X \cong \pi_{(p/1), !} \text{pr}^* \pi_{\Gamma_s}^* X \cong \pi_{(p/1), !} \pi_{(p/1)}^* X \cong \pi_{\omega_1, !} \pi_{\omega_1}^* X \cong \coprod X$$

The last step is to compute $(1, 1)^* \Xi X$, which takes a little more work. In order to identify it, we modify the construction of the cofibration morphism Ξ up to isomorphism. Instead of immediately applying $p_!: \mathbb{D}^{\Gamma_s} \rightarrow \mathbb{D}^{[1]}$, we note that Γ_s is nearly $\omega \times [1]$, if we think of this latter category as

$$\begin{array}{cccccc} -1_0 & 0_0 & 1_0 & 2_0 & 3_0 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & \cdots \end{array}$$

Let $i: \Gamma_s \rightarrow \omega \times [1]$ be the inclusion as indicated by the labelling in the diagram. The category $\omega \times [1]$ has the canonical projection to $[1]$ which we denote $p_{[1]}$. Then we have a factorisation $p = p_{[1]}i$, so we have an isomorphism $p_! \cong p_{[1],!}i_!$. The functor i is a cosieve, so the left Kan extension $i_!$ is extension by zero, and the morphism $p_{[1],1}$ is similar to $p_!$ in that it computes the infinite coproduct of the top row and the bottom row separately. We do not need to make this computation, but in essence we have

$$0^*p_{[1],!}i_!\pi_{\Gamma_s}^* \cong 0 \sqcup X \sqcup X \sqcup \dots \cong \coprod X, \quad 1^*p_{[1],!}i_!\pi_{\Gamma_s}^* \cong X \sqcup X \sqcup \dots \cong \coprod X$$

This reorganising allows us to make explicit the $[1]$ -dimension of Γ_s and reverse the order of the infinite coproduct and the pushout. More specifically, the following diagram commutes up to isomorphism as all the morphisms involved are cocontinuous and involve distinct dimensions of the overall diagram:

$$\begin{array}{ccc} \mathbb{D}^{\omega \times [1]} & \xrightarrow{\text{id}_\omega^* \times i_{\Gamma,!}i_{[1],*}} & \mathbb{D}_{\text{cof}}^\omega \\ p_{[1],!} \downarrow & & \downarrow \coprod \\ \mathbb{D}^{[1]} & \xrightarrow{i_{\Gamma,!}i_{[1],*}} & \mathbb{D}_{\text{cof}} \end{array}$$

Therefore to compute $(1,1)^*\Xi X$, it suffices to compute the following:

$$(1,1)^*i_{\Gamma,!}i_{[1],*}p_!\pi_{\Gamma_s}^* X \cong (1,1)^*i_{\Gamma,!}i_{[1],*}p_{[1],!}i_!\pi_{\Gamma_s}^* X \cong (1,1)^* \coprod (\text{id}_\omega^* \times i_{\Gamma,!}i_{[1],*})i_!\pi_{\Gamma_s}^* X$$

We can commute the coproduct morphism with $(1,1)^*$, so the final challenge is to understand the $(1,1)$ entry of each of the cocartesian squares in $\mathbb{D}_{\text{cof}}^\omega$ that we obtain before taking the infinite coproduct. We have already made the computation of $i_!\pi_{\Gamma_s}^* X \in \mathbb{D}^{\omega \times [1]}$ and found that it is

$$\begin{array}{cccccc} 0 & X & X & X & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ X & X & X & X & \dots \end{array}$$

where all but the first map are (coherent) identities. The pushouts are accomplished independently of each other, which means $(\text{id}_\omega^* \times i_{\Gamma,!}i_{[1],*})i_!\pi_{\Gamma_s}^* X$ has the form (where we rotate all our arrows from vertical to horizontal before pushing out)

$$\begin{array}{cccccc} 0 \rightarrow X & X \rightarrow X & X \rightarrow X & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 \rightarrow X & 0 \rightarrow 0 & 0 \rightarrow 0 & \dots \end{array}$$

So we see that $(1,1)^* \coprod$ applied to this diagram yields a coproduct of X with countably many zero objects, so we conclude that $(1,1)^*\Xi X \cong X$.

By additivity, on K-theory we have $K(T) \simeq K(S) \sqcup K(Q)$. This yields for us $K(\mathbb{I}) \simeq K(\mathbb{I}) \sqcup K(\text{id}_{\mathbb{D}})$, so that $0 \simeq K(\text{id}_{\mathbb{D}})$. The only way for the identity to be equal to zero is if $K(\mathbb{D})$ itself is zero, completing the proof. \square

We now proceed in the spirit of Waldhausen in [Wal85], beginning first with the delooping of the K-theory space $K(\mathbb{D})$. In order to do so, we need a construction of relative K-theory.

To begin, we note that the \mathbf{S}_\bullet -construction on derivators not only gives us a simplicial category, but in fact a simplicial left pointed derivator.

Proposition 6.2. Let $j: [n-1] \rightarrow \text{Ar}[n]$ denote the inclusion $i \mapsto (0, i+1)$. Then $j^*: \mathbf{S}_n\mathbb{D} \rightarrow \mathbb{D}([n-1])$ is an equivalence of categories. Thus we can give $\mathbf{S}_n\mathbb{D} \subset \mathbb{D}^{\text{Ar}[n]}$ the structure of a left pointed derivator via the equivalence with $\mathbb{D}^{[n-1]}$.

Proof. We will construct the quasiinverse directly.

First, consider the functor $i_0: [n-1] \rightarrow [n]$ defined by $i \mapsto i+1$. This map is a cosieve, so the morphism $i_{0,!}$ is extension by zero.

Second, consider the subcategory $D \subset \text{Ar}[n]$ which contains the top row $(0, i)$ as well as all diagonal entries (i, i) . The inclusion $i_1: [n] \rightarrow J$ is a sieve, so $i_{1,*}$ is extension by zero.

The last step is to take the inclusion $i_2: D \rightarrow \text{Ar}[n]$ and compute $i_{2,!}$. We claim that the image of this composition lies in $\mathbf{S}_n\mathbb{D}(e) \subset \mathbb{D}(\text{Ar}[n])$. To see this, we use [Gro13, Proposition 3.10] to detect cocartesian squares, whose statement we provide now (using notation more convenient for our purposes).

Proposition 6.3. Let $\iota: \square \rightarrow K$ be a fully faithful functor and let $u: J \rightarrow K$ be any functor. We may consider the full category $K \setminus \iota(1, 1)$ obtained by removing the image of the bottom-right corner of the square, and then form the comma category

$$\begin{array}{ccc} (K \setminus \iota(1, 1)/\iota(1, 1)) & \longrightarrow & K \setminus \iota(1, 1) \\ \downarrow & \swarrow & \downarrow \\ e & \xrightarrow{\iota(1, 1)} & K \end{array}$$

where the righthand vertical map is the inclusion of the subcategory. This comma category receives a functor from Γ induced by ι , which we denote $\bar{\iota}$.

Assume that $\bar{\iota}: \Gamma \rightarrow (K \setminus \iota(1, 1)/\iota(1, 1))$ admits a left adjoint and that $\iota(1, 1)$ does not lie in the image of $u: J \rightarrow K$. Then for all $X \in \mathbb{D}(J)$, the square $\iota^*u_!X \in \mathbb{D}(\square)$ is cocartesian.

We will apply this ‘detection lemma’ with $J = [n-1]$ and $K = \text{Ar}[n]$. Let $\iota_{i,j}: \square \rightarrow \text{Ar}[n]$ be the square given by $(a, b) \mapsto (i+b, j+a)$ for $0 \leq i < j \leq n-1$, where the ‘flip’ of a and b comes in because of an unfortunate inconsistency in

the notation between \square and $\text{Ar}[n]$. It suffices to prove each of these squares is cocartesian, as any other square will be a composite of such squares. If a larger square can be subdivided into cocartesian squares, then it too is cocartesian by [Gro13, Proposition 3.13(1)].

Because $\text{Ar}[n]$ is a poset, we can identify the comma category

$$(\text{Ar}[n] \setminus (i+1, j+1) / (i+1, j+1))$$

as the full subcategory of $\text{Ar}[n]$ on (p, q) admitting a map to $(i+1, j+1)$, i.e. $p \leq i+1$ and $q \leq j+1$, excluding $(p, q) = (i+1, j+1)$ as it has been removed. Call the resulting category $B^{i,j}$. We now construct a left adjoint for $\iota_{i,j}$ directly. We define $\ell: B^{i,j} \rightarrow \Gamma$ by

$$\ell(p, q) = \begin{cases} (0, 0) & p \leq i \text{ and } q \leq j \\ (0, 1) & q = j+1 \\ (1, 0) & p = i+1 \end{cases}$$

Direct computation shows that $\text{Hom}(\ell(p, q), (a, b)) = \text{Hom}((p, q), \iota_{i,j}(a, b))$ for any elements $(p, q) \in B^{i,j}$ and $(a, b) \in \Gamma$. We can also construct the unit and the counit directly; the counit is just the identity on Γ , and the unit can only be the unique map $(p, q) \rightarrow \iota(\ell(p, q))$ at each $(p, q) \in B^{i,j}$.

This proves that $\iota_{i,j}^* i_{2,!} X$ is a cocartesian square for any $X \in \mathbb{D}(D)$. Pasting these squares together shows that any square in $i_{2,!} X$ is cocartesian. Moreover, if we have $X = i_{1,*} i_{0,!} Y$ for some $Y \in \mathbb{D}([n-1])$, then $(i, i)^* X = 0$ by construction. Therefore $X \in \mathbf{S}_n \mathbb{D}(e)$. Call this total morphism $\Phi: \mathbb{D}([n-1]) \rightarrow \mathbf{S}_n \mathbb{D}$.

Because Φ is constructed as left and right Kan extensions of fully faithful functors, it too is fully faithful. Moreover, it is left adjoint to j^* , with the counit $\text{id}_{\mathbb{D}([n-1])} = j^* \Phi$ the identity modification. Because the left adjoint morphism is fully faithful, the unit is an isomorphism. Because both the unit and counit are invertible modifications, this gives an equivalence of categories. \square

Remark 6.4. We can give an alternative proof that $\mathbf{S}_n \mathbb{D}$ has the structure of a left pointed derivator. First, for $K \in \mathbf{Dir}_f$, we define $\mathbf{S}_n \mathbb{D}(K) \subset \mathbb{D}^{\text{Ar}[n]}(K)$ to be the full subcategory on objects X such that $k^* X \in \mathbf{S}_n \mathbb{D}$ for any $k \in K$. This makes $\mathbf{S}_n \mathbb{D}$ a prederivator on \mathbf{Dir}_f . Der1, Der2, and Der5 follow immediately from its definition as a certain levelwise subcategories of a derivator, and it is also (weakly) pointed because the 0 object of $\mathbb{D}(\text{Ar}[n])$ is in $\mathbf{S}_n \mathbb{D}(e)$.

For the remainder of the axioms, it suffices to show that the left and right Kan extensions in $\mathbb{D}^{\text{Ar}[n]}$ land in the appropriate subcategory. Let $X \in \mathbf{S}_n \mathbb{D}(J)$ and let $u: J \rightarrow K$ be a functor. We only know for sure that $u_! X \in \mathbb{D}^{\text{Ar}[n]}(K)$, so we need to check that for all $k \in K$, $(i, i)^* k^* u_! X = 0$ for all $i \in [n]$ and for all squares $\iota: \square \rightarrow \text{Ar}[n]$, $\iota^* k^* u_! X \in \mathbb{D}^\square$ is cocartesian.

For the first point, let us just examine $(i, i)^*u_!X \in \mathbb{D}(K)$. Because $(i, i)^*$ is cocontinuous, we have $(i, i)^*u_!X \cong u_!(i, i)^*X$. We know that $(i, i)^*X = 0 \in \mathbb{D}(J)$ because $X \in \mathbf{S}_n\mathbb{D}(J)$, so we have $u_!(i, i)^*X = 0 \in \mathbb{D}(K)$ because $u_!$ is a pointed morphism. Therefore $k^*(i, i)^*u_!X = 0 \in \mathbb{D}(e)$, and these first two morphisms commute because they are pullback morphisms in unrelated diagrammatic directions, giving us $(i, i)^*k^*u_!X = 0$ for any $i \in [n]$ and $k \in K$.

For the second point, we have

$$\iota^*k^*u_!X \cong k^*\iota^*u_!X \cong k^*u_!\iota^*X$$

for reasons identical to the above. We know that ι^*X is a cocartesian square in $\mathbb{D}(\square \times J)$, and $u_!$ preserves cocartesian squares. This implies that each $k^*u_!\iota^*X$ is cocartesian in $\mathbb{D}(\square)$, and following the chain of isomorphisms backwards finishes the proof.

There is no difference if we consider the right Kan extension sieve $u: J \rightarrow K$, as the extension by zero morphism $u_*: \mathbb{D}^{\text{Ar}[n]}(J) \rightarrow \mathbb{D}^{\text{Ar}[n]}(K)$ is still cocontinuous. That was the only fact we used in the case of left Kan extensions $u_!$, and so we complete the proof.

Because $\mathbf{S}_n\mathbb{D}$ has the structure of a left pointed derivator, it means that $\mathbf{S}_\bullet\mathbb{D}$ is actually a simplicial object in left pointed derivators. This means we can iterate the \mathbf{S}_\bullet construction, and will do so shortly. But before that, we will define our relative K-theory construction. To do so, we need the following general simplicial constructions.

For any simplicial set Y , we may define a new simplicial set PY (of paths in Y) by precomposing Y by the functor $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}}$ which sends $[n]$ to $[n+1]$ via $i \mapsto i+1$.

Lemma 6.5 (Lemma 1.5.1, [Wal85]). PY is simplicially homotopy equivalent to the constant simplicial set on Y_0 .

There is a projection map $PY \rightarrow Y$ induced by the 0-face map. Moreover, there is a functor $Y_1 \rightarrow PY$ which is the inclusion of 0-simplices, as $(PY)_0 = Y_1$. This gives a sequence $Y_1 \rightarrow PY \rightarrow Y$ for any simplicial set Y .

Now suppose that $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ is a strict cocontinuous morphism of left pointed derivators. We then define the simplicial category $\mathbf{S}_\bullet\Phi$ by the following 2-pullback in \mathbf{CAT} , sometimes called the iso-comma, which lies between the lax pullback and the strict pullback:

$$\begin{array}{ccc} \mathbf{S}_\bullet\Phi & \longrightarrow & P\mathbf{S}_\bullet\mathbb{E} \\ \downarrow & \cong \nearrow & \downarrow d_0 \\ \mathbf{S}_\bullet\mathbb{D} & \xrightarrow{\Phi} & \mathbf{S}_\bullet\mathbb{E} \end{array}$$

Specifically, at each $[n] \in \Delta^{\text{op}}$, we have a square

$$\begin{array}{ccc} \mathbf{S}_n \Phi & \longrightarrow & (PS_{\bullet} \mathbb{E})_n = \mathbf{S}_{n+1} \mathbb{E} \\ \downarrow & \cong \nearrow & \downarrow d_0 \\ \mathbf{S}_n \mathbb{D} & \xrightarrow{\Phi} & \mathbf{S}_n \mathbb{E} \end{array}$$

in which the top-left corner is explicitly the following: an object in $\mathbf{S}_n \Phi$ is a pair $(A \in \mathbf{S}_n \mathbb{D}, B \in \mathbf{S}_{n+1} \mathbb{E})$ along with an isomorphism $f_{A,B}: \Phi(A) \rightarrow d_0(B) \in \mathbf{S}_n \mathbb{E}$. Note that if Φ is not a strict morphism, then there are naturality problems with the face and degeneracy maps of $\mathbf{S}_{\bullet} \Phi$.

However, if Φ is not a strict morphism, then by Proposition 4.1 there are two strict (cocontinuous) morphisms $\Pi_{\Phi}: \tilde{\mathbb{D}} \rightarrow \mathbb{D}$ and $\tilde{\Phi}: \tilde{\mathbb{D}} \rightarrow \mathbb{E}$ such that Π_{Φ} is a weak equivalence and $\Phi \Pi_{\Phi} = \tilde{\Phi}$. While $\mathbf{S}_{\bullet} \Phi$ will not be defined directly, it will have the same homotopy type as $\mathbf{S}_{\bullet} \tilde{\Phi}$. Therefore it is not an issue for us to assume the morphism Φ is strict for the rest of this argument.

In Waldhausen K-theory, $\mathbf{S}_{\bullet} F$ for $F: \mathcal{C} \rightarrow \mathcal{D}$ an exact morphism of Waldhausen categories is again a simplicial Waldhausen category. For us, it is not immediately clear that $\mathbf{S}_{\bullet} \Phi$ should be a simplicial left pointed derivator, so we prove that now.

Proposition 6.6. The simplicial category $\mathbf{S}_{\bullet} \Phi$ underlies a simplicial object in left pointed derivators (which we give the same name).

Proof. For $K \in \mathbf{Dir}_{\mathbf{f}}$, the category $\mathbf{S}_n \Phi(K)$ will have objects a triple

$$A \in \mathbf{S}_n \mathbb{D}(K), B \in \mathbf{S}_{n+1} \mathbb{E}(K), f_{A,B}: \Phi(A) \xrightarrow{\cong} d_0(B),$$

which we will shorten to $(A, B, f_{A,B})$. For $u: J \rightarrow K$, we define $u^*(A, B, f_{A,B})$ to be the triple $(u^*A, u^*B, g_{A,B})$, where $g_{A,B}$ is the map filling in the commutative diagram of isomorphisms below:

$$\begin{array}{ccc} u^* \Phi(A) & \xrightarrow{u^* f_{A,B}} & u^* d_0(B) \\ \gamma_u^{\Phi} \downarrow & & \downarrow \gamma_u^{d_0} \\ \Phi(u^*A) & \xrightarrow{g_{A,B}} & d_0(u^*B) \end{array}$$

The vertical isomorphisms are actually equalities because Φ is assumed to be strict (and d_0 is in any case), so $g_{A,B} = u^* f_{A,B}$. We include the full picture for analogy with what follows. This proves that $\mathbf{S}_n \Phi$ has the structure of a prederivator.

A fair question at this point is why we need the flexibility of an isomorphism $f_{A,B}$ if Φ is assumed to be strict. If we required $f_{A,B}$ to be the identity, then we know that $u^*(f_{A,B})$ would also be the identity by strict 2-functoriality. The issue arises for the left and right Kan extensions, which we address now. Suppose now that

$(A, B, f_{A,B}) \in \mathbf{S}_n\Phi(J)$. Then we define $u_!(A, B, f_{A,B})$ to be $(u_!A, u_!B, h_{A,B})$, where $h_{A,B}$ is the map filling in a similar commutative diagram of isomorphisms:

$$\begin{array}{ccc} u_!\Phi(A) & \xrightarrow{u_!f_{A,B}} & u_!d_0(B) \\ (\gamma_u^\Phi)_! \downarrow & & \downarrow (\gamma_u^{d_0})_! \\ \Phi(u_!A) & \xrightarrow[h_{A,B}]{} & d_0(u_!B) \end{array}$$

Because Φ and d_0 are both cocontinuous, the left mates of the structure isomorphisms γ_u are isomorphisms (by definition). However, even though $\gamma_u^{d_0}$ may be the identity, there is no reason to believe that its mates are also identities; they are only guaranteed to be isomorphisms. Therefore we have

$$h_{A,B} = (\gamma_u^{d_0})_! \circ u_!f_{A,B} \circ (\gamma_u^\Phi)_!^{-1}$$

This explains the definition of $\mathbf{S}_\bullet\Phi$ as an iso-comma (simplicial) category instead of a strict pullback as in Waldhausen's original construction.

These left Kan extensions for $\mathbf{S}_n\Phi$ are natural and (moreover) are the only ones that makes sense. The construction of left Kan extensions also generalises to right Kan extensions along sieves $u: J \rightarrow K$, as the right mates $(\gamma_u^\Phi)_*$ and $(\gamma_u^{d_0})_*$ will also be isomorphisms by Der4R.

For strongness, because the (partial) underlying diagram functors for $\mathbf{S}_n\Phi$ are constructed as a pullback of those for $\mathbf{S}_n\mathbb{D}$ and $\mathbf{S}_{n+1}\mathbb{E}$, they are still full and essentially surjective on finite free categories.

Finally, $\mathbf{S}_n\Phi(e)$ is pointed by $(0, 0, \cong)$, where the isomorphism is unique, which gives $\mathbf{S}_\bullet\Phi$ the structure of a simplicial left pointed derivator. \square

We can now formulate the statement of relative derivator K-theory. Let $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ be a strict cocontinuous morphism of left pointed derivators. First, there is an inclusion $\mathbf{S}_1\mathbb{E} \rightarrow PS_\bullet\mathbb{E}$ as zero simplices, where we view $\mathbf{S}_1\mathbb{E}$ as a constant simplicial left pointed derivator. Composing this with the map $d_0: PS_\bullet\mathbb{E} \rightarrow \mathbf{S}_\bullet\mathbb{E}$ we obtain a sequence

$$\mathbf{S}_1\mathbb{E} \rightarrow PS_\bullet\mathbb{E} \rightarrow \mathbf{S}_\bullet\mathbb{E}$$

the composition of which is constant. We can lift the map $\mathbf{S}_1\mathbb{E} \rightarrow PS_\bullet\mathbb{E}$ to $\mathbf{S}_1\mathbb{E} \rightarrow \mathbf{S}_\bullet\Phi$ using the pullback defining $\mathbf{S}_\bullet\Phi$ and the constant (at zero)

map $\mathbf{S}_1\mathbb{E} \rightarrow \mathbf{S}_\bullet\mathbb{D} \rightarrow \mathbf{S}_\bullet\mathbb{E}$

$$\begin{array}{ccccc}
 \mathbf{S}_1\mathbb{E} & & & & \\
 \downarrow \text{0} & \searrow \text{0-simplices} & & & \\
 & \mathbf{S}_\bullet\Phi & \longrightarrow & P\mathbf{S}_\bullet\mathbb{E} & \\
 & \downarrow \cong \nearrow & & \downarrow d_0 & \\
 & \mathbf{S}_\bullet\mathbb{D} & \xrightarrow{\Phi} & \mathbf{S}_\bullet\mathbb{E} &
 \end{array}$$

Composing with the projection from $\mathbf{S}_\bullet\Phi$ to $\mathbf{S}_\bullet\mathbb{D}$ we obtain a sequence

$$\mathbf{S}_1\mathbb{E} \rightarrow \mathbf{S}_\bullet\Phi \rightarrow \mathbf{S}_\bullet\mathbb{D}$$

the composition of which is again constant. Iterating the \mathbf{S}_\bullet construction, we have the following theorem.

Theorem 6.7. The sequence

$$(6.8) \quad i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \rightarrow i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi \rightarrow i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}$$

is a homotopy fibration after (diagonally) geometrically realizing nerves and passing to the homotopy category of spaces \mathcal{S} .

Proof. We proceed as in [Wal85, Proposition 1.5.5]. We use the following ‘realization lemma’ from [Wal78]:

Lemma 6.9. Let $X_{\bullet\bullet} \rightarrow Y_{\bullet\bullet} \rightarrow Z_{\bullet\bullet}$ be a sequence of bisimplicial sets so that $X_{\bullet\bullet} \rightarrow Z_{\bullet\bullet}$ is constant. Suppose that $X_{\bullet n} \rightarrow Y_{\bullet n} \rightarrow Z_{\bullet n}$ is a homotopy fibration for every n . Suppose further that $Z_{\bullet n}$ is connected for every n . Then the original sequence is also a homotopy fibration.

We are indeed in this situation, up to a little unpacking. We have a sequence of bisimplicial categories, which we will turn into a sequence of trisimplicial sets by taking the nerve:

$$N_\bullet i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \rightarrow N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi \rightarrow N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}.$$

However, let us treat this as a bisimplicial set by considering the first two simplicial directions as one diagonal direction, i.e. if we let \bullet and \star denote the two different directions, we have

$$N_\bullet i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \rightarrow N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\star\Phi \rightarrow N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\star\mathbb{D}$$

which is now a sequence of bisimplicial sets. The first term appears the same because $\mathbf{S}_1\mathbb{E}$ was constant in the \star direction. As geometric realization may be taken variable-by-variable or diagonally (in fact, these give homeomorphic spaces), it suffices to show that this second sequence of bisimplicial sets is a homotopy fibration.

The last thing to check is that $N_\bullet i\mathbf{S}_\bullet \mathbf{S}_n \mathbb{D}$ is connected for all n . But $N_0 i\mathbf{S}_0 \mathbf{S}_n \mathbb{D}$ consists of the zero objects in $\mathbf{S}_n \mathbb{D}(e)$, all of which are isomorphic, hence there is a 1-simplex in $N_1 i\mathbf{S}_0 \mathbf{S}_n \mathbb{D}$ which is this isomorphism. Applying a degeneracy map in the \mathbf{S}_0 -direction will give us a 1-simplex in the diagonal simplicial set $N_1 i\mathbf{S}_1 \mathbf{S}_n \mathbb{D}$ which connects these 0-simplices, showing that this simplicial set is indeed connected.

Therefore let us fix an n and consider the sequence

$$i\mathbf{S}_\bullet \mathbf{S}_1 \mathbb{E} \rightarrow i\mathbf{S}_\bullet \mathbf{S}_n \Phi \rightarrow i\mathbf{S}_\bullet \mathbf{S}_n \mathbb{D}$$

of simplicial left pointed derivators. We will make our argument here and pass to the nerve and the corresponding diagonal simplicial sets as we outlined above.

We will show, as Waldhausen does, that this relative K-theory sequence is homotopic to the trivial homotopy fibration. We will do so using the additivity theorem.

We define a cofibration morphism of derivators

$$\Xi_n : \mathbf{S}_n \Phi \rightarrow (\mathbf{S}_n \Phi)_{\text{cof}}$$

such that $(0, 0)^* \Xi_n$ takes values in a copy of $\mathbf{S}_1 \mathbb{E}$ inside $\mathbf{S}_n \Phi$, $(1, 0)^* \Xi_n = \text{id}_{\mathbf{S}_n \Phi}$, and $(1, 1)^* \Xi_n$ takes values in a copy of $\mathbf{S}_n \mathbb{D}$ inside $\mathbf{S}_n \Phi$.

The sketch for the construction of Ξ_n is the following: for $(A, B, f_{A,B}) \in \mathbf{S}_n \Phi$,

$$\Xi_n(A, B, f_{A,B}) = \begin{array}{ccc} (0, s_n \cdots s_1(0 \rightarrow (0, 1)^* B \rightarrow 0), \cong) & \longrightarrow & (A, B, f_{A,B}) \\ \downarrow & & \downarrow \\ (0, 0, \cong) & \longrightarrow & (A, s_0 d_0(B), f_{A,B}) \end{array}$$

where s_i are the degeneracy maps of the simplicial set $P\mathbf{S}_n \mathbb{E}$. The entry in the top left is a degenerate n -simplex in $P\mathbf{S}_n \mathbb{E}$ which comes from $(0 \rightarrow (0, 1)^* B \rightarrow 0) \in P\mathbf{S}_0 \mathbb{E}$. The isomorphisms between zero objects on the lefthand side are more subtle than they appear, and we will address this below.

To begin with the $\mathbf{S}_n \mathbb{D}$ component of $\mathbf{S}_n \Phi$, we define a cofibration morphism $\mathbf{S}_n \mathbb{D} \rightarrow (\mathbf{S}_n \mathbb{D})_{\text{cof}}$. Consider the map $p_n : \text{Ar}[n] \times \square \rightarrow \text{Ar}[n]$ defined as follows: for $(a, b) = (1, 0)$ or $(1, 1)$, we let $p_n(i, j, a, b) = (i, j)$. Further, we let $p_n(i, j, 0, 0) = (0, 0)$ and $p_n(i, j, 0, 1) = (0, 0)$ be constant. We illustrate the functor $p_2 : \text{Ar}[2] \times \square \rightarrow \text{Ar}[2]$, using bold arrows for the \square dimension of the diagram. We label the objects of the domain according to where they map in the codomain, which shows better how the

pullback p_2^* behaves:

$$\begin{array}{ccc}
\begin{array}{ccc} 0_1 & \rightarrow & 0_1 & \rightarrow & 0_1 \\ & & \downarrow & & \downarrow \\ & & 0_1 & \rightarrow & 0_1 \\ & & & & \downarrow \\ & & & & 0_1 \\ & \Downarrow & & & \\ 0_1 & \rightarrow & 0_1 & \rightarrow & 0_1 \\ & & \downarrow & & \downarrow \\ & & 0_1 & \rightarrow & 0_1 \\ & & & & \downarrow \\ & & & & 0_1 \end{array} & \Longrightarrow & \begin{array}{ccc} 0_1 & \rightarrow & a & \rightarrow & b \\ & & \downarrow & & \downarrow \\ & & 0_2 & \rightarrow & c \\ & & & & \downarrow \\ & & & & 0_3 \\ & \Downarrow & & & \\ 0_1 & \rightarrow & a & \rightarrow & b \\ & & \downarrow & & \downarrow \\ & & 0_2 & \rightarrow & c \\ & & & & \downarrow \\ & & & & 0_3 \end{array} & \rightarrow & \begin{array}{ccc} 0_1 & \rightarrow & a & \rightarrow & b \\ & & \downarrow & & \downarrow \\ & & 0_2 & \rightarrow & c \\ & & & & \downarrow \\ & & & & 0_3 \end{array}
\end{array}$$

The horizontal arrows in the \square dimension are necessarily zero maps, and the vertical arrows are identity maps, as $p_n(-, -, 1, -): \text{Ar}[n] \times [1] \rightarrow \text{Ar}[n]$ defining the righthand vertical map is just the projection $\text{id}_{\text{Ar}[n]} \times \pi_{[1]}$ and $p_n(-, -, 0, -)$ defining the lefthand vertical map is the constant functor $\text{Ar}[n] \times [1] \rightarrow e$. This square is cocartesian, and establishes the construction for $\mathbf{S}_n\mathbb{D}$. Note that the definition of p_n implicitly uses that $n \geq 1$, but for the case $n = 0$, $\text{Ar}[0] = e$ so $p_0: \square \rightarrow e$ is the constant functor π_\square by necessity.

For the $P\mathbf{S}_n\mathbb{E} = \mathbf{S}_{n+1}\mathbb{E}$ component of the derivator $\mathbf{S}_n\Phi$, we define a map $q_n: \text{Ar}[n+1] \times \square \rightarrow \text{Ar}[n+1]$ computing what we want. First, we will have $q_n(i, j, 1, 0) = (i, j)$, just as p_n was defined. To deal with the other $(a, b) \in \square$, we start with $(a, b) = (0, 0)$:

$$q_n(i, j, 0, 0) = \begin{cases} (0, 0) & (i, j) = (0, 0) \\ (0, 1) & i = 0, 1 \text{ and } j \geq 1 \\ (1, 1) & \text{otherwise} \end{cases}$$

We want $q_n(-, -, 0, 1)$ to be a constant functor (as $p_n(-, -, 0, 1)$ was) but in this case we let $q_n(i, j, 0, 1) = (1, 1)$. Finally, for the case $(a, b) = (1, 1)$,

$$q_n(i, j, 1, 1) = \begin{cases} (1, 1) & (i, j) = (0, 0) \\ (1, j) & i = 0 \text{ and } j \geq 1 \\ (i, j) & \text{otherwise} \end{cases}$$

To illustrate q_2 , we have the following picture, where we again label the zeroes:

$$\begin{array}{ccc}
 \begin{array}{c}
 0'_0 \rightarrow \alpha \xrightarrow{=} \alpha \xrightarrow{=} \alpha \\
 \downarrow \quad \downarrow \quad \downarrow \\
 0'_1 \rightarrow 0'_1 \rightarrow 0'_1 \\
 \quad \quad \downarrow \quad \downarrow \\
 \quad \quad 0'_1 \rightarrow 0'_1 \\
 \quad \quad \quad \downarrow \\
 \quad \quad \quad 0'_1 \\
 \Downarrow
 \end{array}
 & \Rightarrow &
 \begin{array}{c}
 0'_0 \rightarrow \alpha \rightarrow \beta \rightarrow \gamma \\
 \downarrow \quad \downarrow \quad \downarrow \\
 0'_1 \rightarrow \delta \rightarrow \varepsilon \\
 \quad \quad \downarrow \quad \downarrow \\
 \quad \quad 0'_2 \rightarrow \zeta \\
 \quad \quad \quad \downarrow \\
 \quad \quad \quad 0'_3 \\
 \Downarrow
 \end{array}
 & \rightarrow &
 \begin{array}{c}
 0'_0 \rightarrow \alpha \rightarrow \beta \rightarrow \gamma \\
 \downarrow \quad \downarrow \quad \downarrow \\
 0'_1 \rightarrow \delta \rightarrow \varepsilon \\
 \quad \quad \downarrow \quad \downarrow \\
 \quad \quad 0'_2 \rightarrow \zeta \\
 \quad \quad \quad \downarrow \\
 \quad \quad \quad 0'_3
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c}
 0'_1 \rightarrow 0'_1 \rightarrow 0'_1 \rightarrow 0'_1 \\
 \downarrow \quad \downarrow \quad \downarrow \\
 0'_1 \rightarrow 0'_1 \rightarrow 0'_1 \\
 \quad \quad \downarrow \quad \downarrow \\
 \quad \quad 0'_1 \rightarrow 0'_1 \\
 \quad \quad \quad \downarrow \\
 \quad \quad \quad 0'_1 \\
 \Downarrow
 \end{array}
 & \Rightarrow &
 \begin{array}{c}
 0'_1 \rightarrow 0'_1 \rightarrow \delta \rightarrow \varepsilon \\
 \downarrow \quad \downarrow = \quad \downarrow = \\
 0'_1 \rightarrow \delta \rightarrow \varepsilon \\
 \quad \quad \downarrow \quad \downarrow \\
 \quad \quad 0'_2 \rightarrow \zeta \\
 \quad \quad \quad \downarrow \\
 \quad \quad \quad 0'_3
 \end{array}
 \end{array}$$

To check that this defines the functor $\Xi_n : \mathbf{S}_n\Phi \rightarrow (\mathbf{S}_n\Phi)_{\text{cof}}$, we need to construct the isomorphisms $\Phi(p_n^*A) \rightarrow d_0(q_n^*B)$ corresponding to $\Xi_n(A, B, f_{A,B})$, and we will demonstrate how to do so at each corner of \square . The isomorphisms at $(1, 0)$ and $(1, 1)$ are just $f_{A,B}$, but the morphisms at $(0, 0)$ and $(0, 1)$ are between specific zero objects. In the above diagrams, $f_{A,B}$ defines an isomorphism $\Phi(0_i) \rightarrow 0'_i$ for each i . By construction, $(0, 0)^*p_n^*A$ and $(0, 1)^*p_n^*A$ are the constant diagram on the zero object 0_1 , and $d_0((0, 0)^*q_n^*B)$ and $d_0((0, 1)^*q_n^*B)$ are the constant diagram on the zero object $0'_1$. Therefore $(0, 0)^*f_{A,B}$ is still the appropriate isomorphism of zero objects and no additional data is required. Indeed, if we were forced to compose isomorphisms of the form $\Phi(0_1) \rightarrow 0'_1 \rightarrow 0'_2$ (or worse, $\Phi(0_1) \rightarrow 0'_1 \leftarrow 0'_0$), we could not complete this construction coherently (as we discussed in Remark 5.18).

Thus we define

$$\Xi_n(A, B, f_{A,B}) = (p_n^*A, q_n^*B, p_n^*f_{A,B}).$$

The structure isomorphisms $\gamma_u^{\Xi_n}$ come directly from the structure isomorphisms $\gamma_u^{p_n}$ and $\gamma_u^{q_n}$.

The source morphism $(0, 0)^*\Xi_n$ has image a subcategory of $\mathbf{S}_n\Phi$ equivalent to $\mathbf{S}_1\mathbb{E}$, and the quotient morphism $(1, 1)^*\Xi_n$ has image a subcategory equivalent to $\mathbf{S}_n\mathbb{D}$. In particular, every object $(A, s_0d_0(B), f_{A,B})$ is isomorphic to the object $(A, s_0\Phi(A), \text{id})$. The target morphism $(1, 0)^*\Xi$ has image precisely $\mathbf{S}_n\Phi$. These morphisms are moreover essentially surjective.

Now, there are two projections $\pi_1: \mathbf{S}_n\Phi \rightarrow \mathbf{S}_1\mathbb{E}$ and $\pi_2: \mathbf{S}_n\Phi \rightarrow \mathbf{S}_n\mathbb{D}$. The first is defined by $\pi_1(A, B, f_{A,B}) = (0 \rightarrow (0, 1)^*B \rightarrow 0)$ and the second defined by $\pi_2(A, B, f_{A,B}) = A$. This gives a total projection

$$\rho: \mathbf{S}_n\Phi \rightarrow \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D}.$$

This map has a section $\sigma: \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \rightarrow \mathbf{S}_n\Phi$ given by

$$\sigma((0 \rightarrow b \rightarrow 0), A) = (A, s_n \cdots s_1(0 \rightarrow b \rightarrow 0) \sqcup s_0\Phi(A), \text{id}).$$

For example, for $A = (a_1 \rightarrow a_2 \rightarrow a_3) \in \mathbf{S}_2\mathbb{D}$, the component of $\sigma(b, A)$ in $P\mathbf{S}_2\mathbb{E}$ is

$$\begin{array}{ccccc} 0 \rightarrow b \rightarrow b \sqcup \Phi(a_1) & \rightarrow & b \sqcup \Phi(a_2) & & \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi(a_1) & \longrightarrow & \Phi(a_2) \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Phi(a_3) \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

The actual construction of σ relies on the additive structure of $\mathbf{S}_n\Phi$. We first define $\mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \rightarrow \mathbf{S}_n\Phi \times \mathbf{S}_n\Phi$ by the two morphisms

$$\mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \rightarrow \mathbf{S}_1\mathbb{E} \xrightarrow{S'} \mathbf{S}_n\Phi \quad \text{and} \quad \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \rightarrow \mathbf{S}_n\mathbb{D} \xrightarrow{Q'} \mathbf{S}_n\Phi$$

where the first map in each case is the projection. The morphism S' is defined by

$$S'(0 \rightarrow b \rightarrow 0) = (0, s_n \cdots s_1(0 \rightarrow b \rightarrow 0), \cong)$$

and the morphism Q' is defined by

$$Q'(A) = (A, s_0\Phi(A), \text{id}).$$

We then compose $(S', Q'): \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \rightarrow \mathbf{S}_n\Phi \times \mathbf{S}_n\Phi$ with the coproduct map $\mathbf{S}_n\Phi \times \mathbf{S}_n\Phi \cong \mathbf{S}_n\Phi^{e \sqcup e} \rightarrow \mathbf{S}_n\Phi$. The definition of this map and how to strictify it is contained in Equation 5.9 above.

By construction, $\rho\sigma \cong \text{id}_{\mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D}}$, and so obtain a homotopy after passing to K-theory. We now to show that the reverse composition $\sigma\rho$ is homotopic to the identity.

By applying the additivity theorem to Ξ_n , the identity on $\mathbf{S}_n\Phi$ is homotopic to the sum of the inclusion of $\mathbf{S}_1\mathbb{E}$ and $\mathbf{S}_n\mathbb{D}$, and this a morphism isomorphic to $\sigma\rho$. Therefore map of sequences

$$\begin{array}{ccccc} i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} & \longrightarrow & i\mathbf{S}_\bullet\mathbf{S}_n\Phi & \longrightarrow & i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D} \\ \downarrow = & & \downarrow \sim & & \downarrow = \\ i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} & \longrightarrow & i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \times i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D} & \longrightarrow & i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D} \end{array}$$

has all vertical maps equivalences. The bottom sequence is a trivial homotopy fibration, so the top sequence is also a homotopy fibration, completing the proof. \square

Definition 6.10. For $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ a strict cocontinuous morphism of left pointed derivators, define

$$K(\Phi) := \Omega^2|i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi|.$$

There is one more corollary of relative K-theory that bears mentioning before using this definition.

Corollary 6.11. The topological space $K(\mathbb{D})$ is an infinite loop space.

Proof. If we take the case $\Phi = \text{id}_{\mathbb{D}}: \mathbb{D} \rightarrow \mathbb{D}$, we can identify $\mathbf{S}_\bullet\Phi$ with $PS_\bullet\mathbb{D}$. There is certainly an equivalence of simplicial categories $\mathbf{S}_\bullet\text{id}_{\mathbb{D}}(K) \rightarrow PS_\bullet\mathbb{D}(K)$ for each $K \in \mathbf{Dir}_{\mathbf{f}}$ as the pullback of the equivalence $\text{id}_{\mathbb{D}}: \mathbf{S}_\bullet\mathbb{D}(K) \rightarrow \mathbf{S}_\bullet\mathbb{D}(K)$ is still an equivalence. But because the morphism $\mathbf{S}_\bullet\text{id}_{\mathbb{D}} \rightarrow PS_\bullet\mathbb{D}$ is defined globally and is levelwise an equivalence of categories, we get an equivalence of simplicial left pointed derivators.

Using this replacement, we have a fibration

$$i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{D} \rightarrow P(i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}) \rightarrow i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D},$$

where P modifies the first simplicial direction. But now the middle term is contractible, giving a homotopy equivalence

$$|i\mathbf{S}_\bullet\mathbb{D}| \cong |i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{D}| \xrightarrow{\sim} \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}|$$

We use here that the bisimplicial category $i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{D}$ is homotopy equivalent to the simplicial category $i\mathbf{S}_\bullet\mathbb{D}$. The equivalence is given (using morphisms of derivators and passing to maps up to homotopy in the geometric realization) by the forgetful functor $\mathbf{S}_1\mathbb{D} \rightarrow \mathbb{D}$ on the one hand and an iterated extension by zero morphism for the inverse.

Specifically, consider the cosieve $t: e \rightarrow [1]$ given by the inclusion of the target and the sieve $s: [1] \rightarrow \text{Ar}[1]$ given by the inclusion into $(0, 0) \rightarrow (0, 1)$. Then the morphism $s_*t_!: \mathbb{D} \rightarrow \mathbb{D}(\text{Ar}[1])$ is, for $a \in \mathbb{D}$,

$$a \mapsto 0 \rightarrow a \mapsto \begin{array}{c} 0 \rightarrow a \\ \downarrow \\ 0 \end{array}$$

This justifies the isomorphism above.

We now replace \mathbb{D} by the simplicial left pointed derivator $\mathbf{S}_\bullet\mathbb{D}$ and repeat the process, obtaining

$$|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}| \cong |i\mathbf{S}_\bullet\mathbf{S}_1\mathbf{S}_\bullet\mathbb{D}| \xrightarrow{\sim} \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}|$$

which implies that $|i\mathbf{S}_\bullet\mathbb{D}|$ is equivalent to $\Omega^2|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}|$.

By induction, we then conclude

$$K(\mathbb{D}) = \Omega|i\mathbf{S}_\bullet\mathbb{D}| \xrightarrow{\sim} \Omega^{(n)}|i\mathbf{S}_\bullet^{(n)}\mathbb{D}|$$

and so $K(\mathbb{D})$ is an infinite loop space. In particular, we can view $K(\mathbb{D})$ as a connective Ω -spectrum. \square

Corollary 6.12. Let $\Phi: \mathbb{D} \rightarrow \mathbb{E}$ be a strict cocontinuous morphism of left pointed derivators. Then there is a homotopy fibration

$$K(\Phi) \rightarrow K(\mathbb{D}) \rightarrow K(\mathbb{E})$$

Proof. If we rotate the fibration sequence of Equation 6.8 to the left twice (and replace $\mathbf{S}_1\mathbb{E}$ by \mathbb{E}), we have a fibration sequence

$$\Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi| \rightarrow \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}| \rightarrow |i\mathbf{S}_\bullet\mathbb{E}|.$$

By the above corollary, $|i\mathbf{S}_\bullet\mathbb{D}| \rightarrow \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}|$ is a homotopy equivalence. Replacing the middle term and applying Ω everywhere, the corollary follows. \square

The next logical step is to use Theorem 6.7 to prove a localization theorem in K-theory and answer (positive or negatively) Maltiniotis' conjecture that Verdier quotients of triangulated derivators get sent to long exact sequences in K-theory. Unfortunately, the necessity of coherent diagrams in derivator K-theory obstructs [Wal85, Theorem 1.6.4] from proceeding verbatim. In particular, that technique being directly translatable would mean that Waldhausen K-theory agrees with derivator K-theory in general, which has been proven false in [TV04] and [MR11]. Nonetheless, future work will address the issue of localization in a novel way which should avoid this obstruction.

As a coda, we can relate derivator K-theory to the K-theory of stable ∞ -categories as described in [BGT13].

Proposition 6.13. Derivator K-theory is an additive invariant of stable ∞ -categories. Specifically, there is a functor $K_{\mathbb{D}}: \mathbf{Cat}_{\infty}^{\text{ex}} \rightarrow \mathcal{S}_{\infty}$ from the category of small stable ∞ -categories and exact functors to the category of spectra that inverts Morita equivalences, preserves filtered colimits, and sends split-exact sequences to fibrations.

Proof. First, how does one obtain a derivator from an ∞ -category? Arlin (né Carlson) in [Arl20] gives the following natural definition [Definition 9] using the quasicategory model for ∞ -categories. For Q a quasicategory, define the prederivator $\text{HO}(Q)$ by

$$\text{HO}(Q): J \mapsto \text{Ho}(Q^{N_\bullet J}), \quad u: J \rightarrow K \mapsto \text{Ho}(N_\bullet u^*): \text{Ho}(Q^{N_\bullet K}) \rightarrow \text{Ho}(Q^{N_\bullet J})$$

where $Q^{N_\bullet J}$ is the quasicategory of simplicial maps $N_\bullet J \rightarrow Q$. The action of $\text{HO}(Q)$ on natural transformations is the only sensible one given the above. Arlin proves

that all such prederivators satisfy Der1, Der2, and Der5. Moreover, if Q admits (homotopy) limits and colimits, so does $\mathrm{HO}(Q)$. In particular, if Q is a stable quasicategory, it admits all homotopy finite limits and colimits, so $\mathrm{HO}(Q)$ is a strong derivator on \mathbf{Dir}_f . Moreover, an exact functor of stable ∞ -categories preserves all finite limits and colimits, so induces a cocontinuous morphism of the corresponding derivators. Therefore $\mathrm{HO}(\mathbf{Cat}_\infty^{\mathrm{ex}}) \subset \mathbf{Der}_K$.

Arlin further proves in [Arl20, Corollary 20] that quasicategories embed simplicially fully-faithfully into the simplicial enrichment of (pre)derivators developed by Muro-Raptis in [MR17], wherein the authors also prove that the derivator K-theory functor $\mathbf{Der}_K \rightarrow \mathcal{S}$ admits a simplicial enrichment in [MR17, Proposition 5.1.3]. We will not reiterate the details of these simplicial enrichment on derivators here, but it can be noted that it requires working with strict morphisms of derivators only. Since these are the only morphisms that pass honestly to K-theory, this is no problem. Part of Arlin's proof is that all functors of quasicategories give strict morphisms of derivators, so we do not have an invisible 'strictification' step in the middle.

As derivator K-theory takes values in infinite loop spaces, i.e. connective spectra, we can postcompose the K-theory functor with the (∞ -categorical) suspension spectrum functor $\Sigma_+^\infty: \mathcal{S} \rightarrow \mathcal{S}_\infty$ without changing anything. The total definition becomes

$$K_{\mathbb{D}}: \mathbf{Cat}_\infty^{\mathrm{ex}} \xrightarrow{\natural} \mathbf{Cat}_\infty^{\mathrm{perf}} \subset \mathbf{QC}at \xrightarrow{\mathrm{HO}} \mathbf{Der}_K \xrightarrow{K} \mathcal{S} \rightarrow \mathcal{S}_\infty$$

We consider all categories above to be simplicial categories (as our model for ∞ -categories). Thus what we have above is not literally a functor between ∞ -categories, but induces one once enough (co)fibrant replacement is incorporated.

The functor \natural is the idempotent completion functor, which also appears as the first step of the universal additive invariant in [BGT13]; this ensures that Morita equivalences are inverted. The category $\mathbf{Cat}_\infty^{\mathrm{perf}}$ is just the full subcategory idempotent complete ∞ -categories. That split exact sequences are sent to fibrations is exactly the additivity theorem.

Finally, we need to address filtered colimits. First, the idempotent completion functor is left adjoint to the inclusion $\mathbf{Cat}_\infty^{\mathrm{perf}} \subset \mathbf{Cat}_\infty^{\mathrm{ex}}$ by (for example) the comments near [BGT13, Definition 2.14], so preserves all colimits. Therefore assume that $Q: I \rightarrow \mathbf{Cat}_\infty^{\mathrm{perf}}$ is a filtered diagram. For any $K \in \mathbf{Dir}_f$, we can consider the comparison map

$$\mathrm{colim}_I \mathrm{HO}(Q_i)(K) \rightarrow \mathrm{HO}(\mathrm{colim}_I Q_i)(K)$$

which, if we unwind the definition, is

$$\mathrm{colim}_I \mathrm{Ho}(Q_i^{N \bullet K}) \rightarrow \mathrm{Ho}\left(\left(\mathrm{colim}_I Q_i\right)^{N \bullet K}\right)$$

The functor $\mathrm{Ho}: \mathbf{Cat}_\infty \rightarrow \mathbf{Cat}$ which gives the underlying category of a quasicategory is left adjoint to the nerve functor, so preserves all colimits. Therefore we need only consider the comparison map between quasicategories

$$\mathrm{colim}_I Q_i^{N_\bullet K} \rightarrow (\mathrm{colim}_I Q_i)^{N_\bullet K}$$

Here we may use the fact that $K \in \mathbf{Dir}_f$, so that $N_\bullet K$ has only finitely many nondegenerate simplices. In particular, this is a compact object in simplicial sets, so the functor $(-)^{N_\bullet K}$ commutes with filtered colimits.

We therefore obtain an equivalence

$$\mathrm{colim}_I \mathrm{HO}(Q_i)(K) \rightarrow \mathrm{HO}(\mathrm{colim}_I Q_i)(K)$$

for all $K \in \mathbf{Dir}_f$, which assemble to an equivalence of the corresponding derivators. This equivalence passes through to K-theory, completing the argument. \square

In light of this proposition, we should expect that the comparison map between Waldhausen and derivator K-theory that Muro-Raptis studied in [MR17] agrees with the ‘universal trace map’ $K_\infty \rightarrow K_{\mathbb{D}}$ guaranteed by the results of [BGT13, Theorem 10.3]. In particular, the spectrum of natural transformations $\mathrm{Nat}(K_\infty, K_{\mathbb{D}})$ is isomorphic to $K_{\mathbb{D}}(\mathcal{S}_\infty^\omega)$, the K-theory associated to the derivator of (compact) spectra, i.e. the derivator K-theory of modules over the sphere spectrum \mathbb{S} .

We know that $\pi_0 K_\infty(\mathbb{S}) \cong \mathbb{Z}$, $\pi_1 K_\infty(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\pi_2 K_\infty(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$, which can be found in [BM19]. Using the results of [Rap19] (building on [Mur08]), the comparison map $K_\infty \rightarrow K_{\mathbb{D}}$ is 2-connected. This means that $\pi_0 K_{\mathbb{D}}(\mathbb{S}) \cong \mathbb{Z}$, so that the derivator trace map should correspond to an integer $x \in \mathbb{Z}$. In [BGT13, Theorem 10.6], the authors prove that the Dennis trace $K \rightarrow THH$ corresponds to $1 \in THH_0(\mathbb{S}) \cong \mathbb{Z}$, but this relies on earlier (classical) computations by Waldhausen. Future work will explore this perspective more closely and identify the integer corresponding to the ‘derivator trace’. In particular, Raptis’ conjecture of whether the derivator trace is generally not an isomorphism on π_2 might start by showing that $\pi_2 K_{\mathbb{D}}(\mathbb{S}) = 0$, though we have no evidence for this at present.

REFERENCES

- [Arl20] Kevin Arlin. A higher Whitehead theorem and the embedding of quasicategories in prederivators. *Homology Homotopy Appl.*, 22(1):117–139, 2020.
- [BGT13] Andrew J. Blumberg, David Gepner, and Gonalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.
- [BM19] Andrew J. Blumberg and Michael A. Mandell. The homotopy groups of the algebraic K-theory of the sphere spectrum. *Geom. Topol.*, 23(1):101–134, 2019.
- [Bor94] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

- [Cis10] Denis-Charles Cisinski. Catégories dérivables. *Bull. Soc. Math. France*, 138(3):317–393, 2010.
- [CN08] Denis-Charles Cisinski and Amnon Neeman. Additivity for derivator K -theory. *Adv. Math.*, 217(4):1381–1475, 2008.
- [Col20] Ian Coley. The theory of half derivators. In preparation, 2020.
- [Fra96] Jens Franke. Uniqueness theorems for certain triangulated categories with an Adams spectral sequence. <http://www.math.uiuc.edu/K-theory/0139/Adams.pdf>, 1996.
- [Gar05] Grigory Garkusha. Systems of diagram categories and K -theory. II. *Math. Z.*, 249(3):641–682, 2005.
- [Gar06] G. Garkusha. Systems of diagram categories and K -theory. I. *Algebra i Analiz*, 18(6):131–186, 2006. <https://arxiv.org/abs/math/0401062>.
- [GPS14] Moritz Groth, Kate Ponto, and Michael Shulman. Mayer-vietoris sequences in stable derivators. *Homology Homotopy Appl.*, 16(1):265–294, 2014.
- [Gra89] Daniel R. Grayson. Exterior power operations on higher K -theory. *K-Theory*, 3(3):247–260, 1989.
- [Gra11] Daniel R. Grayson. The additivity theorem in algebraic K -theory. *Doc. Math.*, 16:457–464, 2011.
- [Gro90] Alexandre Grothendieck. Les dérivateurs. <https://webusers.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html>, 1990.
- [Gro13] Moritz Groth. Derivators, pointed derivators and stable derivators. *Algebr. Geom. Topol.*, 13(1):313–374, 2013.
- [Hel88] Alex Heller. Homotopy theories. *Mem. Amer. Math. Soc.*, 71(383):vi+78, 1988.
- [LN17] Ioannis Lagkas-Nikolos. Levelwise modules over separable monads on stable derivators. *Journal of Pure and Applied Algebra*, 2017.
- [Mal07] Georges Maltsiniotis. La K -théorie d’un dérivateur triangulé. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 341–368. Amer. Math. Soc., Providence, RI, 2007.
- [MR11] Fernando Muro and George Raptis. A note on K -theory and triangulated derivators. *Adv. Math.*, 227(5):1827–1845, 2011.
- [MR17] Fernando Muro and George Raptis. K -theory of derivators revisited. *Ann. K-Theory*, 2(2):303–340, 2017.
- [Mur08] Fernando Muro. Maltsiniotis’s first conjecture for k_1 . *International Mathematics Research Notices*, 2008:rnm153, 2008.
- [Nee97a] Amnon Neeman. K -theory for triangulated categories. I(A). Homological functors. *Asian J. Math.*, 1(2):330–417, 1997.
- [Nee97b] Amnon Neeman. K -theory for triangulated categories. I(B). Homological functors. *Asian J. Math.*, 1(3):435–529, 1997.
- [Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.
- [Qui73] Daniel Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
- [Rap19] George Raptis. Higher homotopy categories, higher derivators, and K -theory. arXiv:1910.04117, 2019.

- [Sch02] Marco Schlichting. A note on K -theory and triangulated categories. *Invent. Math.*, 150(1):111–116, 2002.
- [TV04] Bertrand Toën and Gabriele Vezzosi. A remark on K -theory and S -categories. *Topology*, 43(4):765–791, 2004.
- [Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.
- [Wal78] Friedhelm Waldhausen. Algebraic K -theory of generalized free products. I, II. *Ann. of Math. (2)*, 108(1):135–204, 1978.
- [Wal85] Friedhelm Waldhausen. Algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
- [Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory.

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