Abelian Powers and Periods

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Outline

Getting started

Definition

Let u be a partial word over a finite alphabet $A \cup \diamond$. Then for any $a \in A$, we define $|u|_a$ to be the number of occurrences of a in u, and we define $|u|_\diamond$ similarly. Note that $\sum_{a \in A \cup \diamond} |u|_a = |u|$.

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Definition

Let w be a full word over a finite alphabet A. We say that w is an abelian pth power if there exists a subword $u = u_0 \cdots u_{p-1}$ of w such that $|u_0|_a = |u_i|_a$ for all $i \in \{1, \ldots, p-1\}$ and all $a \in A$.

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Definition

If w is not an abelian pth power, then we say w is abelian p-free or w avoids abelian pth powers.

A useful lemma

Lemma

[1] Let $p > 1$ be an integer, and let $v_0 \cdots v_{p-1}$ be a partial word over a k-letter alphabet $A = \{a_0, \ldots, a_{k-1}\}$ such that $|v_i| = |v_0|$, for all i. Let $d_i = \max_j |v_j|_{a_i}$, for $0 \leq i < k$. Then $v_0 \cdots v_{p-1}$ is an abelian pth power if and only if $d_0 + \cdots + d_{k-1} \leq |v_0|$.

Examples

Let $w_0 = abcbcabac$, $w_1 = abccbbabc$, and $w_2 = \diamond bcabcb\diamond a$.

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w_2 is an abelian cube since $w_0 \uparrow abcabcbca$, an abelian cube.

The problem

A sample construction

[1] There exists an infinite word w over $A = \{a, b\}$ which avoids abelian 4th powers. Let c be any letter not in A. Let $k_i = 5 \cdot 6^i$. Then we can define an infinite partial word w' over $A \cup \{c\}$ by

$$w'(j) = \begin{cases} \diamond & \text{if } j = k_i \text{ for some } i \\ c & \text{if } j = k_i + 1 \text{ or } j = k_i - 1 \text{ for some } i \\ w(j) & \text{otherwise} \end{cases}$$

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In every construction of infinite partial words with infinitely many holes avoiding abelian pth powers for any p, holes are always spaced at least exponentially. The question was posed as to whether holes could ever be spaced 'closer than exponentially.'

New definitions

Definition

Let w be a partial word. We define the hole function of w to be the unique strictly increasing function $H : \mathbb{N} \to \mathbb{N}$ such that $w[H(i) - 1] = \diamond$ is the ith hole of w and if $w[j] = \diamond$, then $j + 1 \in \text{Im}(H)$.

New definitions

Definition

Let w be a full word avoiding abelian pth powers over a finite alphabet. We define the hole density function of w by $d_w(n) : \mathbb{N} \to \mathbb{N}_0$. Formally, $d_w(n) = \max\{\gamma_{v_i} : |v_i| = n, v_i \in \text{Sub}(w)\}$, where $\gamma_v = \max\{|v'_\diamond|_\diamond\}$ such that $v'_\diamond$ contains all the letters of v in order with arbitrarily many holes inserted between any pair of letters or before or after v and $v'_\diamond$ still avoids abelian pth powers.

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Definition

Let v be a subword of w, where w avoids abelian pth powers. If $v'_\diamond$ is formed by inserting holes into v and $|v'_\diamond|_\diamond = d_w(|v|)$, then we say v has maximum hole density and let $v_\diamond = v'_\diamond$.

New definitions

Eschewing obfuscation...

Given a word w, $d_w(n)$ tells us how many holes we can stick into a length n subword without losing abelian p-freeness, and if $|v| = n$, $v_\diamond$ and $|v_\diamond|_\diamond = d_w(n)$ is one of the most-holey subwords.

Illustration

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Let $v_0 = abcb$. Then $(v_0)_\diamond = \diamond abcb$ is the only way to add holes to v_0 without creating a non-trivial abelian square. However, v_0 does not have maximum hole density. Let $v_1 = abcd$, and note $(v_1)_\diamond = \diamond abcd\diamond$, which does attain maximum hole density.

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We might have written $(v_1)_\diamond = \diamond abc\diamond d$, which shows that the construction with maximum hole number is not unique.

Hole density and you

Proposition

Let w be a full word over a finite alphabet A avoiding abelian pth powers and let d_w be its hole density function. Then:

(a) $d_w(n_0 + n_1) \leq d_w(n_0) + d_w(n_1)$

(b) $d_w(k \cdot n) \leq k \cdot d_w(n)$ *for all* $k, n \in \mathbb{N}$

(c) *If* $p > 2$, $d_w(n+1) \leq d_w(n) + (p-2)$

(d) *If* $p = 2$, $d_w(n+1) \leq d_w(n) + 1$

(e) *If* $d_w(N) = 0$ *for some* N, *then* $d_w(n) = 0$ *for all* $n > N$.

Hole density and you

Proof (sketch)

(a) $d_w(n_0 + n_1) \leq d_w(n_0) + d_w(n_1)$

(b) $d_w(k \cdot n) \leq k \cdot d_w(n)$ for all $k, n \in \mathbb{N}$

For (a), assume we have $d_w(n_0 + n_1) > d_w(n_0) + d_w(n_1)$. Then there exists v such that $|v| = n_0 + n_1$ and $v_\diamond$ such that $|v_\diamond| > d_w(n_0 + n_1)$. Then we can find either a subword with n_0 or n_1 non-hole letters containing more than $d_w\,(n_0)$ or $d_w\,(n_1)$ holes, respectively.

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(b) follows immediately from (a) by letting $n_0 = n_1 = n$ and proceeding by induction.

Hole density and you

Proof (sketch)

(c) If $p > 2$, $d_w(n+1) \leq d_w(n) + (p-2)$

(d) If $p = 2$, $d_w(n+1) \leq d_w(n) + 1$

The proofs for (d) and (e) both rely on the fact that a word cannot have too many holes in a row. It is not difficult to see that adding more than $p-2$ holes in a step causes a contradiction, as well as adding 2 or more holes in the abelian square case.

Hole density and you

Proof (sketch)

(e) If $d_w(N) = 0$ for some N, then $d_w(n) = 0$ for all $n > N$.

In essence, a word cannot 'come back from the grave' and suddenly start being able to accept holes once it hits zero. For if it could, then there exists and abelian p-free subword v of w of length $N + 1$ containing some number of holes. But v itself contains a subword of length N with at least one hole that also avoids abelian pth powers, which contradictions $d_w(N) = 0$ □

Bounds

Now that we have a better sense of the hole density function, we return to an important corollary about hole spacing and abelian pth powers.

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Now that we have a better sense of the hole density function, we return to an important corollary about hole spacing and abelian pth powers.

Corollary

[1] Let w be a partial word with infinitely many holes over a finite alphabet, and let $p > 1, m > 0$ be integers. Assume there are fewer than m letters between each pair of consecutive holes in w. Then w contains an abelian pth power.

Bounds

Proposition

Let w be a full word avoiding abelian pth powers over a finite alphabet with hole density function d_w. Then $d_w(n) \notin \Omega(n)$.

Bounds

Proof (sketch)

Let w be a full word avoiding abelian pth powers over a finite alphabet with hole density function d_w. Then $d_w(n) \notin \Omega(n)$.

Assume $d_w(n) \in \Omega(n)$. Then there exists $\alpha > 0$ and $N \in \mathbb{N}$ such that $\alpha \cdot n \leq d_w(n)$ for all $n > N$.

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Let w be a full word avoiding abelian pth powers over a finite alphabet with hole density function d_w. Then $d_w(n) \notin \Omega(n)$.

Assume $d_w(n) \in \Omega(n)$. Then there exists $\alpha > 0$ and $N \in \mathbb{N}$ such that $\alpha \cdot n \leq d_w(n)$ for all $n > N$.

By assumption, for all $m \in \mathbb{N}$, $\alpha \cdot (N+m) \leq d_w(N+m)$. If we add $m = \lceil \frac{1}{\alpha} \rceil$ letters to a subword of length N, then the maximum hole density increases by at least one, since

$$1 + \alpha \cdot N \leq \alpha \cdot (N+m) \leq d_w(N+m)$$

Bounds

Proof (sketch)

Let w be a full word avoiding abelian pth powers over a finite alphabet with hole density function d_w. Then $d_w(n) \notin \Omega(n)$.

Thus, let v be a subword of w with $|v| > N$ and maximum hole density and let $v_\diamond$ be a corresponding partial word. Then

$$H(i) - H(i-1) < N + \left\lceil \frac{1}{\alpha} \right\rceil$$

for any such $v_\diamond$ and for all i.

Bounds

Proof (sketch)

Let w be a full word avoiding abelian pth powers over a finite alphabet with hole density function d_w. Then $d_w(n) \notin \Omega(n)$.

Hence there is an integer that bounds the distance between successive holes, and by the earlier corollary, $v_\diamond$ is an abelian pth power, which is a contradiction. $\square$

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Corollary

If w is a full word avoiding abelian pth powers over a finite alphabet with maximum hole density d_w, then $d_w(n) \in o(n)$, i.e. for any $\alpha > 0$, there exists $N \in \mathbb{N}$ such that $d_w(n) \leq \alpha \cdot n$ for all $n > N$.

Bounds

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Theorem

Let w be an infinite full word avoiding abelian pth powers and let $w'_\diamond$ be an infinite partial word with infinitely many holes, not necessarily equal to $w_\diamond$ the maximal case. Let H be the hole function of $w'_\diamond$ and d_w be the hole density function of w. If $d_w(n) \in O(f(n))$, then $H(i) \in \Omega\left(f^{-1}(i)\right)$.

Bounds

Proof (sketch)

If $d_w(n) \in O\left(f(n)\right)$, then $H\left(i\right) \in \Omega\left(f^{-1}(i)\right)$.

Given a subword $v_\diamond$ of $w'_\diamond$ with n letters, we know $|v_\diamond|_\diamond \leq d_w(n)$. Let $|v_\diamond|_\diamond = j$ and assume $v_\diamond = w\left[0..H(j) - 1\right]$. Then

$$j = |v_\diamond|_\diamond \leq d_w\left(H(j)\right) \leq d_w\left(|v_\diamond|\right)$$

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$$j = |v_\diamond|_\diamond \leq d_w\left(H(j)\right) \leq d_w\left(|v_\diamond|\right)$$

There exists $\beta > 0$ and $N \in \mathbb{N}$ such that $d_w(n) \leq \beta \cdot f(n)$ for $n > N$.

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Proof (sketch)

If $d_w(n) \in O\left(f(n)\right)$, then $H\left(i\right) \in \Omega\left(f^{-1}(i)\right)$.

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$$j = |v_\diamond|_\diamond \leq d_w\left(H(j)\right) \leq d_w\left(|v_\diamond|\right)$$

There exists $\beta > 0$ and $N \in \mathbb{N}$ such that $d_w(n) \leq \beta \cdot f(n)$ for $n > N$.

Additionally, if $d_w(n) \in O\left(f(n)\right)$, then $d_w^{-1}(i) \in \Omega\left(f^{-1}(i)\right)$, where $d_w^{-1}(i) = \max\left\{n : d_w(n) = i\right\}$.

Bounds

Proof (sketch)

If $d_w(n) \in O(f(n))$, then $H(i) \in \Omega\left(f^{-1}(i)\right)$.

Since its inverse is bounded above, there exists γ and $N \in \mathbb{N}$ such that $d_w^{-1}(i) \geq \gamma \cdot f^{-1}(i)$ for $i > N$. So

$$d_w\left(H(j)\right) \geq j \implies H(j) \geq d_w^{-1}(j) \geq \gamma \cdot f^{-1}(j)$$

where the first inequality holds for all j and last inequality only holds if $j > N$, which is sufficient to show $H(i) \in \Omega\left(f^{-1}(i)\right)$. $\square$

Bounds

Corollary

Let w be an infinite full word avoiding abelian pth powers, and $w_\diamond$ be a corresponding infinite partial word with infinitely many holes with hole function H. Then $H(i) \in \omega(i)$.

The corollary follows directly from $d_w(n) \in o(n)$ and the preceding proposition.

Powers that be

Definition

A word w over an alphabet A has abelian period p if $w = u_0 u_1 u_2 \cdots u_m u_{m+1}$, where $m \geq 1$, $|u_1| = |u_2| = \cdots = |u_m| = p$, $|u_0| > 0$, and $|u_0|_a \leq |u_1|_a = |u_2|_a = \cdots = |u_m|_a \geq |u_{m+1}|_a$ for all $a \in A$.

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Definition

A partial word w over an alphabet A has abelian period p if $w = u_0 u_1 u_2 \cdots u_m u_{m+1}$, where $m \geq 1$, $|u_1| = |u_2| = \cdots = |u_m| = p$, $|u_0| > 0$, and there exists a full word v over A, $|v| = p$, such that $|u_i|_a \leq |v|_a$ for all $i \in \{0, \ldots, m+1\}$ and all $a \in A$.

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Note that now we are dealing with finite words that always are abelian powers. Also note the similarity of the second definition to the earlier lemma from [1]. This intuition will be useful.

Definition and example

Definition

Let $u_0 u_1 \cdots u_{m+1}$ and $v_0 v_1 \cdots v_{n+1}$ be factorizations of a partial word w into abelian periods p and q, respectively, with $p < q$. We say that the periods p and q match up if the equality $u_0 u_1 \cdots u_i = v_0 v_1 \cdots v_j$ holds for some integers $i, j \leq m$.

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[2] Let $w = abaaabaaabaaababaaaaabbaaaab$. Then w has abelian periods 4 and 6 which do not match up, and we write
$ab.aaab.aaab.aaab.abaa.aaab.baaa.ab$ and
$abaaa|baaaba|aababa|aaaabb|aaaab$, or (most commonly) combined as
$ab.aaa|b.aaab.a|aab.aba|a.aaab.b|aaa.ab$

Background

A powerful theorem by Constantinescu and Ilie was the motivation for much of the work in [2].

Theorem

[3] If a word w has abelian periods p and q which are relatively prime and $|w| \geq 2pq - 1$, then w has period $\gcd(p, q) = 1$.

That is, that if a word is long enough with coprime periods, it must be unary.

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That is, that if a word is long enough with coprime periods, it must be unary.

Further, in [2], it was proven that this bound is optimal. However, when approaching the case when $\gcd(p, q) > 1$, a key lemma had made an incorrect assumption, which needed correcting.

The problem

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If a word w has abelian periods p and q with $\gcd(p,q) = d$, $d > 1$, then w has at most cardinality d.

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This conjecture is false.

Damning evidence!

$$cbba.abd|acb.aabbcd.|aabbcd.abc|abd.aabbc$$

satisfies the assumptions of the conjecture for $p = 6$, $q = 9$, but has cardinality $\gcd(p, q) + 1 = 4$.

The problem

Lemma

For a word w with abelian periods p and q such that $\gcd(p, q) = d$, $d > 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

The problem

Nearly True Lemma

For a word w with abelian periods p and q such that $\gcd(p, q) = d$, $d > 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

Further damning evidence!

$$abaabaaba.aba|abaaba.abaabaaba.abaaba|aba.aba$$

has $p = 9$, $q = 21$, $u_0 = abaabaaba$, and $u_1 = abaabaabaaba$. The periods do not meet though $||u_0| - |v_0|| = gcd(21, 9) = 3$.

The solution

For the remainder of this section, we assume that all words are full and that p, q are integers satisfying $p < q$, $\gcd(p, q) = d$, $d > 1$, and $p = dp'$, $q = dq'$ (we can assume that $p' > 1$). Here $u_0 u_1 \cdots u_{m+1}$ and $v_0 v_1 \cdots v_{n+1}$ are factorizations of w into abelian periods p and q, respectively.

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We will need this lemma from number theory.

Lemma

Let $a, b \in \mathbb{N}$ be two coprime integers, and without loss of generality assume $1 < a < b$. Then for all $0 \leq \mu < b$, there exist $s, t \in \mathbb{N}$ such that $0 \leq s < b$, $0 \leq t < a$, and $sa - tb = \mu$.

The solution

Proof

Let $a, b \in \mathbb{N}$ be two coprime integers, and without loss of generality assume $1 < a < b$. Then for all $0 \leq \mu < b$, there exist $s, t \in \mathbb{N}$ such that $0 \leq s < b$, $0 \leq t < a$, and $sa - tb = \mu$.

By the Euclidean Algorithm, there exist s_0 and t_0 such that

$$s_0 a - t_0 b = 1 = \gcd(a, b)$$

with $|s_0| < b$ and $|t_0| < a$.

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If $s_0, t_0 < 0$, then let $s = s_0 + b$ and $t = t_0 + a$ so that $s, t > 0$. Equality is preserved because

$$s_0 a - t_0 b = (s + b) \cdot a - (t + a) \cdot b = sa - tb + ba - ab = sa + tb$$

If $s_0, t_0 > 0$, let $s = s_0$ and $t = t_0$.

The solution

Proof

Let $a, b \in \mathbb{N}$ be two coprime integers, and without loss of generality assume $1 < a < b$. Then for all $0 \leq \mu < b$, there exist $s, t \in \mathbb{N}$ such that $0 \leq s < b$, $0 \leq t < a$, and $sa - tb = \mu$.

Simply by multiplying the base case,

$$(ns) \cdot a - (nt) \cdot b = n$$

If $ns < b$ and $nt < a$, then we are done.

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If $ns \geq b$ and $nt \geq a$, then let $s' = ns - b$ and $t' = nt - a$ and notice

$$s'a - t'b = (ns - b) \cdot a - (nt - a) \cdot b = (ns) \cdot a - (nt) \cdot b - ba + ab = (ns) \cdot a - (nt) \cdot b$$

Again, if $s' < b$ and $t' < a$, we are done.

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If $s' \geq b$ and $t' \geq a$, let $s'' = s' - b$ and $t'' = t' - a$, and so on.

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If $s' \geq b$ and $t' \geq a$, let $s'' = s' - b$ and $t'' = t' - a$, and so on.

Assume for some point in this process that $s^{(i)} < b$ but $t^{(i)} \geq a$. Then since $-t^{(i)}b \leq -ab$,

$$n = s^{(i)}a - t^{(i)}b \leq s^{(i)}a - ab < ba - ab = 0 < n$$

which is a contradiction.

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Let $a, b \in \mathbb{N}$ be two coprime integers, and without loss of generality assume $1 < a < b$. Then for all $0 \leq \mu < b$, there exist $s, t \in \mathbb{N}$ such that $0 \leq s < b$, $0 \leq t < a$, and $sa - tb = \mu$.

On the other hand, if we have $s^{(i)} \geq b$ but $t^{(i)} < a$, then $s^{(i)}a \geq ba$, and so

$$n = s^{(i)}a - t^{(i)}b \geq ba - t^{(i)}b \geq ba - (a-1) \cdot b = b > n$$

Thus the case where only one of s and t is out of bounds is impossible. Because we may always reduce $(ns) \cdot a - (nt) \cdot b$, the lemma follows. $\square$

Remark and example

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Let $sa - tb = 1$ as in the previous lemma. If $s', n < b, t' < a$, and $s'a - t'b = n$, then $s' = ns \mod b$ and $t' = nt \mod a$.

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Example

Let $a = 3$ and $b = 7$. Then we have $s_0 = -2$ and $t_0 = -1$ to give $-2 \cdot 3 - (-1) \cdot 7 = 1$. Set $s = -2 + 7 = 5$ and $t = -1 + 3 = 2$, so that $5 \cdot 3 - 2 \cdot 7 = 1$.

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Let $a = 3$ and $b = 7$. Then we have $s_0 = -2$ and $t_0 = -1$ to give $-2 \cdot 3 - (-1) \cdot 7 = 1$. Set $s = -2 + 7 = 5$ and $t = -1 + 3 = 2$, so that $5 \cdot 3 - 2 \cdot 7 = 1$. Let $\mu = 5$. Then

$$\begin{aligned} 25 \cdot 3 - 10 \cdot 7 = 5 &\to (25 - 7) \cdot 3 - (10 - 3) \cdot 7 \\ &\to (18 - 7) \cdot 3 - (7 - 3) \cdot 7 \\ &\to (11 - 7) \cdot 3 - (4 - 3) \cdot 7 = 4 \cdot 3 - 1 \cdot 7 = 5 \end{aligned}$$

where $s' = 4 = 25 \mod 7$ and $t' = 1 = 10 \mod 3$ satisfy the conditions of the lemma.

Our new lemma

Now we are in a position to correct the lemma. The amendment is in purple.

Lemma

For a word w with abelian periods p and q such that $\gcd(p,q) = d$, $d > 1$, *and* $|w| \geq \text{lcm}(p,q) - 1$, *$p$ and q match up if and only if* $||u_0| - |v_0|| = \mu d$ *for some integer* $\mu \geq 0$.

Our new lemma

Proof (sketch)

For a word w with abelian periods p and q such that $\gcd(p, q) = d$, $d > 1$, and $|w| \geq \text{lcm}(p, q) - 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

Suppose that $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$. Since our argument does not depend on which length is greater, we may assume that $|v_0| \geq |u_0|$.

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Consider the subword $w' = u_1 \cdots u_{m+1} = u'_0 u'_1 \cdots u'_{m+1}$ where $u'_0 = \varepsilon$, $u'_1 = u_1, \ldots, u'_{m+1} = u_{m+1}$, and $w' = v'_0 v'_1 \cdots v'_{n+1}$ where $v'_0 = v_0[|u_0|..|v_0|)$, $v'_1 = v_1, \ldots, v'_{n+1} = v_{n+1}$.

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Note that $|v_0'| = \mu d$ and since $|v_0'| < |v_0| \leq q = q'd$, we get that $0 \leq \mu < q'$. Thus, periods p and q match up if there exist non-negative integers s and t such that $sp = \mu d + tq$, where the s p-periods of the matching end at length $sp = d(sp')$ and the t q-periods of the matching end at length $\mu d + tq = d \cdot (\mu + tq')$.

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The lengths sp and $\mu d + tq$ are equal when sp' and $\mu + tq'$ are equal, which is possible for any $0 \leq \mu < q'$ by our new lemma. However, as our damning evidence from above showed, we must deal with the length of our word w.

Our new lemma

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For a word w with abelian periods p and q such that $\gcd(p,q) = d$, $d > 1$, and $|w| \geq \mathrm{lcm}(p,q) - 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

The maximum of $|w|$ is found by maximizing $|u_0|$ and $|v_0|$. From the prior lemma, we see the length before the first matching is $\mu d + sp$ or $\mu d + tq$, depending on whether $|u_0|$ or $|v_0|$ is larger.

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Because $s \leq q - 1$ and $t \leq p - 1$, if we choose μ such that $q' - p' = \mu$ then we can achieve the maximum for both $|u_0|$ and $|v_0|$. Additionally, under this circumstance we have
$(q' - 1) \cdot p = q'p - p = p'q - q + \mu d = (p' - 1) \cdot q + \mu d$, as required.

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For a word w with abelian periods p and q such that $\gcd(p, q) = d$, $d > 1$, and $|w| \geq \operatorname{lcm}(p, q) - 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

Therefore the longest length before p and q match is

$$|v_0| + tq = |u_0| + sp = (p - 1) + (q' - 1) \cdot p = \operatorname{lcm}(p, q) - 1$$

which proves the backwards direction of the lemma.

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Therefore the longest length before p and q match is

$$|v_0| + tq = |u_0| + sp = (p-1) + (q'-1) \cdot p = \text{lcm}(p,q) - 1$$

which proves the backwards direction of the lemma.

For the forwards direction, $||u_0| - |v_0|| \neq \mu d$ for any $0 \leq \mu < q'$. The property p meets q is equivalent to $|u_0| + sp = |v_0| + tq$, where we restrict s and t as in the premise, reformulated to $sp - tq = |v_0| - |u_0|$.

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For a word w with abelian periods p and q such that $\gcd(p, q) = d$, $d > 1$, and $|w| \geq \text{lcm}(p, q) - 1$, p and q match up if and only if $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$.

Then since $d|p$ and $d|q$, any linear combination of p and q will also be divisible by d. So $|v_0| - |u_0| = sp - tq \equiv 0 \mod d$, which contradicts our assumption that $d \nmid ||u_0| - |v_0||$. Therefore the lemma holds in both directions. $\square$

And all is well!

With these two lemmas in place, we have the following corollary.

Corollary

If a word w has abelian periods p and q with $\gcd(p, q) = d$, $d > 1$, $|w| \geq 2\,\mathrm{lcm}(p, q) - 1$, and $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$, then the abelian periods p and q have at least two matchings.

The BIG finale!

Now we may conclude what [2] set out to do.

Theorem

If a word w has abelian periods p and q with $\gcd(p, q) = d$, $d > 1$, and $||u_0| - |v_0|| = \mu d$ for some integer $\mu \geq 0$, then w has at most cardinality d for $|w| \geq 2\,\mathrm{lcm}(p, q) - 1$.

We omit the proof due to (probably) lack of time and scope.

What now?

Abelian periods

The issues surrounding abelian periods seem fairly put to rest, pending that the corrective lemmas and proofs are not incorrect.

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Abelian powers

However, there are still many open question surrounding sub-exponential hole spacing and the hole density function.

Conjectures

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Let w be an infinite partial word avoiding abelian pth powers, and let H be its hole function. Then $H(i) \in \omega(i)$.

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We know this is possible for any infinite partial word arising from an infinite full word. However, at least in the finite case, not all abelian p-free partial words arise from abelian p-free full words.

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Let w be an infinite partial word avoiding abelian pth powers, and let H be its hole function. Then $H(i) \in \omega(i)$.

Why?

We know this is possible for any infinite partial word arising from an infinite full word. However, at least in the finite case, not all abelian p-free partial words arise from abelian p-free full words. For example:

$$u = abccbabac \rightarrow u_\diamond = abccb\diamond abac$$

where u contains an abelian cube and $u_\diamond$ does not. We have not examined the hole density function on general words containing abelian powers.

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Computer testing has shown that nearly all the time $d_w(n+1) \geq d_w(n) - 1$. However, this was shown to be false for a certain N in the fixed point of Pleasants' abelian square-free morphism on five letters in [4].

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Additionally, computer tests have not yet shown a case where $d_w(n+1) > d_w(n) + 1$. However, in the general case, the proof is yet illusive.

Conjectures

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Let u be an infinite binary word avoiding abelian 4th powers, v be an infinite tertiary word avoiding abelian cubes, and w be an infinite quinary word avoiding abelian squares. Then $d_u(n), d_v(n), d_w(n) \in O(\log(n))$.

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It remains an open question whether an infinite partial words with infinitely many holes avoiding abelian 4th powers (respectively cubes) exists at all over a binary (respectively tertiary) alphabet. It seems unlikely, then, that we would be able to construct such a word with anything less than exponential hole spacing.

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For the w case, we would like to say quaternary instead of quinary, but [1] proved that for a quaternary word avoiding abelian squares, $d_w(n) \in O(n)$. Thus in the quinary case, we conjecture that sub-exponential hole spacing is impossible.

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Polynomial hole spacing is possible via full words, that is, there exists some word w over a k-letter alphabet avoiding abelian pth powers such that $d_w(n) \in O(\log(n))$.

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Why?

It should work, right?

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A World Wide Web server interface has been established at `www.uncg.edu/cmp/research/abelianrepetitions4` for automated use of the program.

THE END